OSCILLATION OF LINEAR MECHANICAL SYSTEMS DUE TO IMPACT EXCITATION

dr. Miha Boltežar

2004

1 Introduction, lecture placement

Linear oscillation of single degree of freedom systems	Free	- undamped - damped (viscous)
	Forced	- harmonic excitation - periodic excitation **- impulse excitation (transient)**
Oscillation of multiple degree of freedom systems	Free ...	
Oscillation of continuous systems		

2 Used methods

- representing the excitation using the FOURIER integral
- **use of convolution integral**
- use of Laplace transform
- approximation of excitation force using an interpolation model and convolution
- numerical integration of the equation of motion

3 The use of convolution integral

3.1 Unit impulse definition

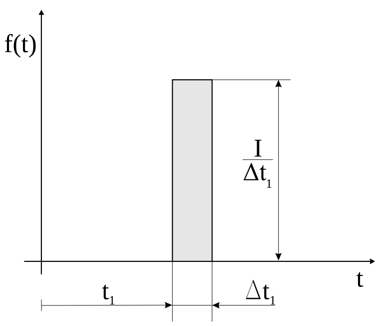


Figure 1: Unit impulse.

Unit impulse; Dirac delta function

$$\Delta t_1 \longrightarrow 0 \Longrightarrow \frac{I}{\Delta t_1} \longrightarrow \text{across all boundaries}$$

$$\delta(t-t_1) = \begin{cases} 0 & \text{for any } t \neq t_1 \\ \infty & \text{for } t = t_1. \end{cases}$$

The integral of the product of Dirac delta function with an arbitrary function $f(t)$ equals:

$$\int_0^\infty f(t)\delta(t-t_1)\mathrm{d}t = f(t), \qquad 0 < t_1 < \infty$$

In the special case, where $f(t) = 1$:

$$\int_0^\infty \delta(t-t_1)\mathrm{d}t = 1, \qquad 0 < t_1 < \infty$$

3.2 Impulse response function

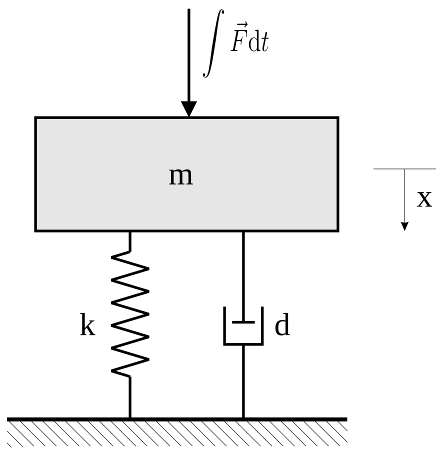


Figure 2: Oscillator.

$$\vec{I} = \Delta\vec{p} = m\vec{v}_2 - m\vec{v}_1,$$

where

$$\vec{I} = \int_0^t \vec{F}(t)\,\mathrm{d}t$$

and

$$\vec{v}_2 = \frac{\vec{I}}{m}$$

If the oscillator in Fig. 2 is excited by the impulse $\vec{I}$, the initial conditions state:

1. $x(0) = 0$
2. $\dot{x}(0) = \frac{I}{m}$

a) **Undamped oscillator**; $d = 0$
The equation of motion for an undamped oscillator depicted in Fig. 2:

$$m\ddot{x} + kx = 0$$

The solution of the equation of motion:

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t), \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

From the initial conditions we can calculate $A = 0$ and $B = \frac{I}{m\omega_0}$, therefore, the response of the system can be rewritten as:

$$x(t) = \frac{I}{m\omega_0}\sin(\omega_0 t)$$

The impulse response function:

$$g(t) = \frac{x(t)}{I} = \frac{1}{m\omega_0}\sin(\omega_0 t)$$

b) **Damped oscillator** The equation of motion for an undamped oscillator depicted in Fig. 2:

$$m\ddot{x} + d\dot{x} + kx = 0$$

The solution of the equation of motion:

$$x(t) = e^{-\delta\omega_0 t}[A\cos(\omega_{0d} t) + B\sin(\omega_{0d} t)]$$

From the initial conditions, the constants A and B can be calculated, which are $A = 0$ and $B = \frac{I}{m\omega_{0d}}$.

$$x(t) = \frac{I}{m\omega_{0d}}e^{-\delta\omega_0 t}\sin(\omega_{0d} t)$$

Impulse response function:

$$g(t) = \frac{1}{m\omega_{0d}}e^{-\delta\omega_0 t}\sin(\omega_{0d} t)$$

4 Convolution integral

Necessary properties of a system to use the convolutional integral:

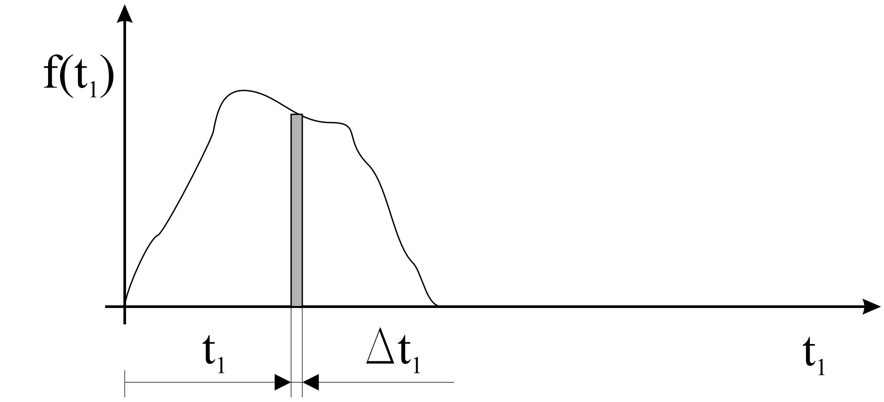
- Linearity
- Time invariance
- Causality

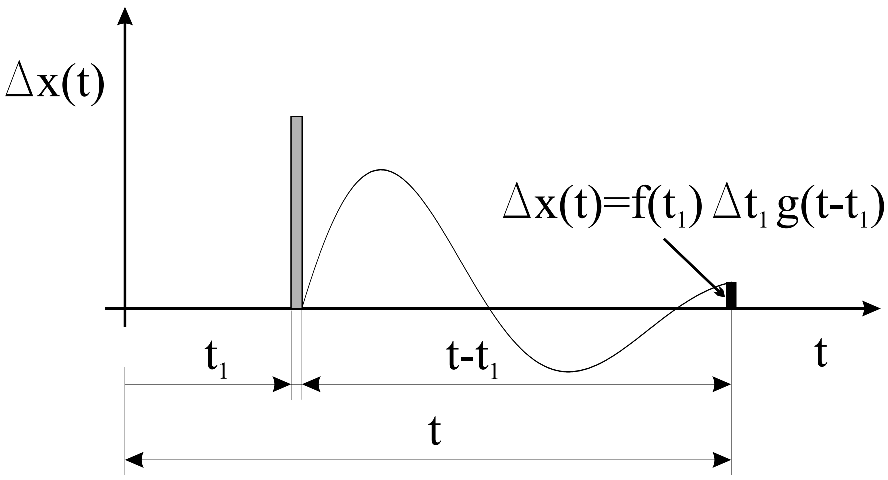
$$\Delta x(t) = f(t_1)\Delta t_1 g(t - t_1)$$

By using the principle of superposition we obtain

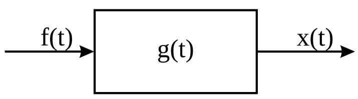
$$x(t) = \lim_{\Delta_i \to 0}\sum_{i=1}^{n} f(t_i)g(t - t_i)\Delta t_i = \int_0^t f(t_1)g(t - t_1)\mathrm{d}t_1$$

BOREL's theorem





4.1 Writing convolution using a memory variable τ



$$x(t) = \int_0^t f(t_1) g(t - t_1) \mathrm{d}t_1$$

More generally, we can write:

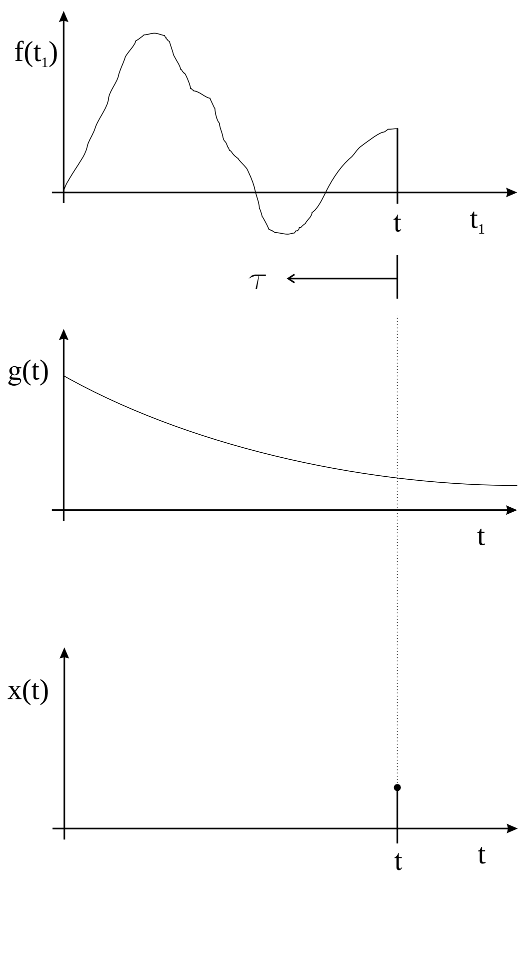
$$x(t) = \int_{-\infty}^t f(t_1) g(t - t_1) \mathrm{d}t_1$$

A new time variable can be introduced $\tau = t - t_1$, $\mathrm{d}\tau = -\mathrm{d}t_1$

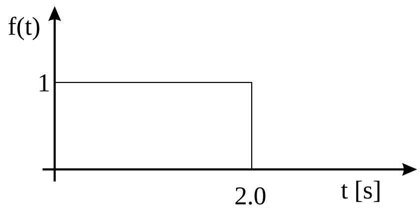
$$x(t) = \int_0^\infty f(t-\tau)g(\tau)\mathrm{d}\tau$$

$$x(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$$

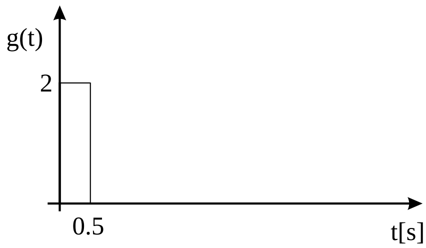
Notation
$x = f * g = g * f$
$x = f \circ g = g \circ f$
τ - memory time variable
$g(\tau)$ - dynamic memory of the system (oscillator)



$$x(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau = \lim_{\Delta\tau_i \to 0} \sum_{i=0}^{n} f(t - i\Delta\tau)g(i\Delta\tau)\Delta\tau$$

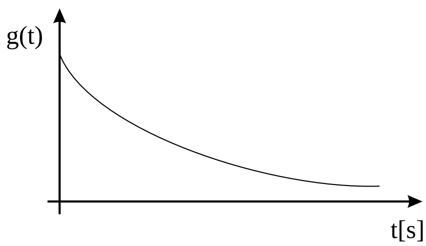


a)

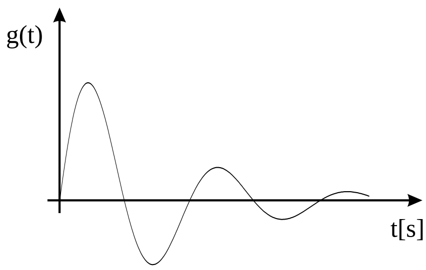


b)

c)



d)



5 Examples

5.1 Example 1

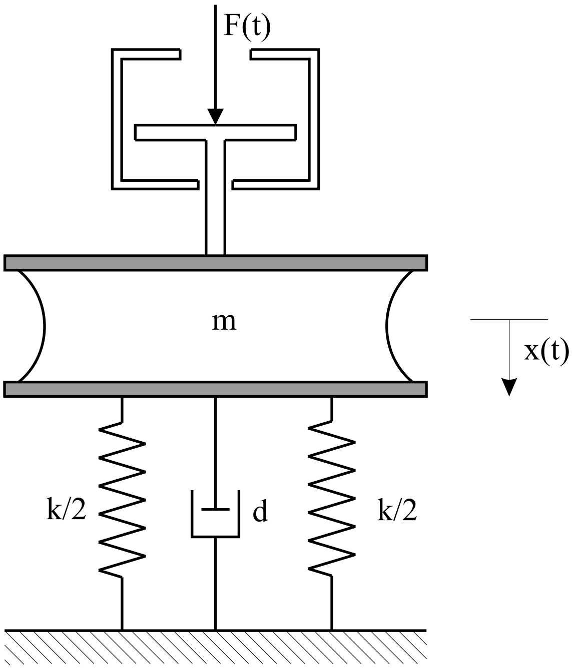


Figure 3: Schematic representation of a press.

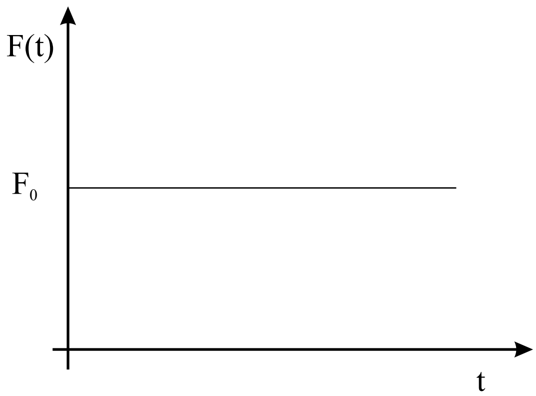


Figure 4: Step force.

Determine the response of the press in Fig. 3 to the step force:

a) for an undamped system and

b) for a damped system

Solution

a) Undamped example

The impulse response function of an undamped oscillator is:

$$g(t) = \frac{1}{m\omega_0}\sin(\omega_0 t) \tag{1}$$

The system response can be obtained by solving the convolution integral:

$$x(t) = \int_0^t f(t_1)g(t-t_1)\mathrm{d}t_1 \tag{2}$$

Prior to solving, the impulse response function (1) is written in the form:

$$g(t-t_1) = \frac{1}{m\omega_0}\sin(\omega_0(t-t_1)) \tag{3}$$

By inserting (3) into (2) and taking into account that $f(t_1) = F_0$, the system response can be calculated:

$$\begin{aligned} x(t) &= \int_0^t \frac{F_0}{m\omega_0}\sin(\omega_0(t-t_1))\mathrm{d}t_1 \\ &= \frac{F_0}{m\omega_0}\left(\frac{1}{\omega_0}\cos(\omega_0(t-t_1))\right)\Bigg|_0^t \\ &= \frac{F_0}{m\omega_0^2}\left(1-\cos(\omega_0 t)\right) \\ &= \frac{F_0}{k}\left(1-\cos(\omega_0 t)\right) \end{aligned} \tag{4}$$

The response is shown in Fig.5.

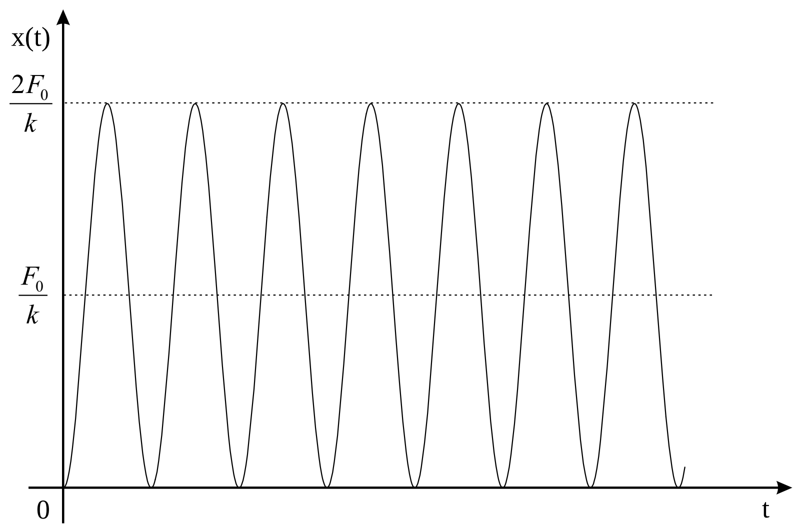


Figure 5: Response of an undamped press.

b) Damped example

The impulse response function of an undamped oscillator is:

$$g(t) = \frac{1}{m\omega_{0d}} e^{-\delta\omega_0 t} \sin(\omega_{0d} t) \tag{5}$$

The system response can be obtained by solving the convolution integral:

$$x(t) = \int_0^t f(t_1) g(t - t_1) \mathrm{d}t_1 \tag{6}$$

The impulse response function (5) is rewritten in the form:

$$g(t - t_1) = \frac{1}{m\omega_{0d}} e^{-\delta\omega_0 (t-t_1)} \sin(\omega_{0d}(t - t_1)) \tag{7}$$

The system response is given by solving the following integral:

$$x(t) = \int_0^t F_0 \frac{1}{m\omega_{0d}} e^{-\delta\omega_0 (t-t_1)} \sin(\omega_{0d}(t - t_1)) \mathrm{d}t_1 \tag{8}$$

Homework: solve the integral in Eq. (8)

The system response is shown in Fig. 6.

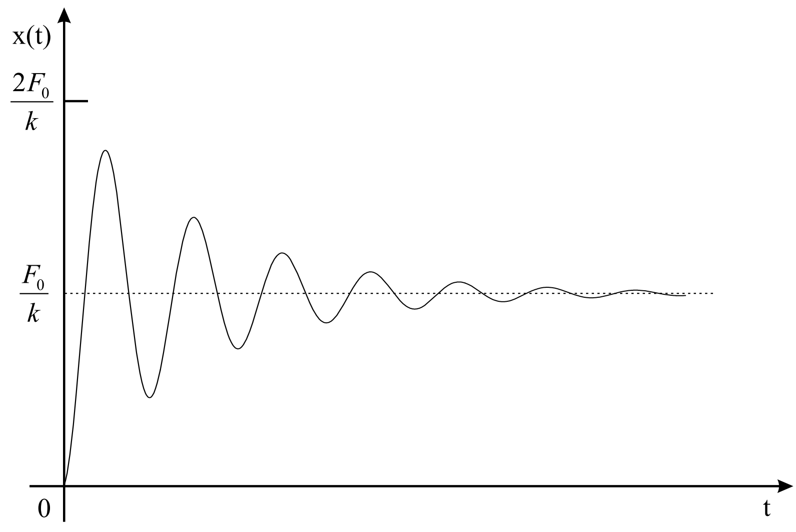


Figure 6: Response of a damped press.

5.2 Example 2

Determine the response of the system in Fig. 7.

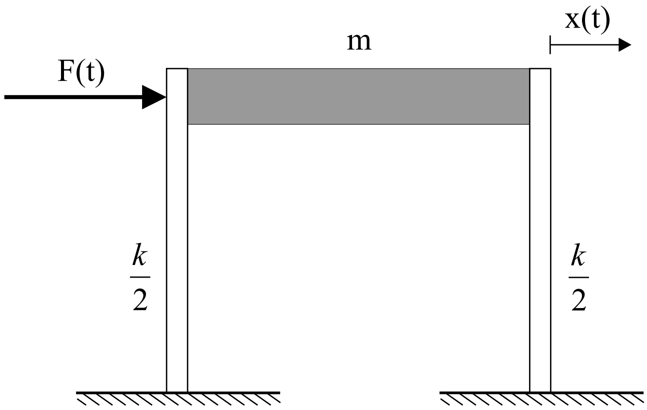


Figure 7: Cantilever structure.

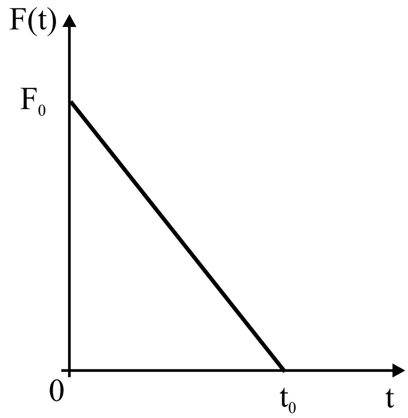


Figure 8: Excitation force.

The force can be written as a function of time:

$$f(t) = \begin{cases} F_0 \left(1 - \frac{t}{t_0}\right) & 0 \leq t \leq t_0 \\ 0 & t > t_0 \end{cases} \tag{9}$$

The impulse response function is written as:

$$g(t - t_1) = \frac{1}{m\omega_0} \sin(\omega_0 (t - t_1)) \tag{10}$$

The system response can be calculated by solving the following integral:

$$x(t) = \int_0^t F_0 \left(1 - \frac{t_1}{t_0}\right) \frac{1}{m\omega_0} \sin(\omega_0 (t - t_1)) \mathrm{d}t_1 \tag{11}$$

The response for $0 \leq t \leq t_0$ is:

$$x(t) = \frac{F_0}{k} \left[1 - \frac{t}{t_0} - \cos(\omega_0) + \frac{1}{\omega_0 t_0} \sin(\omega_0 t) \right] \tag{12}$$

The response after the force is no longer present i.e. when $t > t_0$, is written below:

$$x(t) = \frac{F_0}{k\omega_0 t_0} [(1 - \cos(\omega_0 t_0)) \sin(\omega_0 t) - (\omega_0 t_0 - \sin(\omega_0 t_0)) \cos(\omega_0 t)] \tag{13}$$

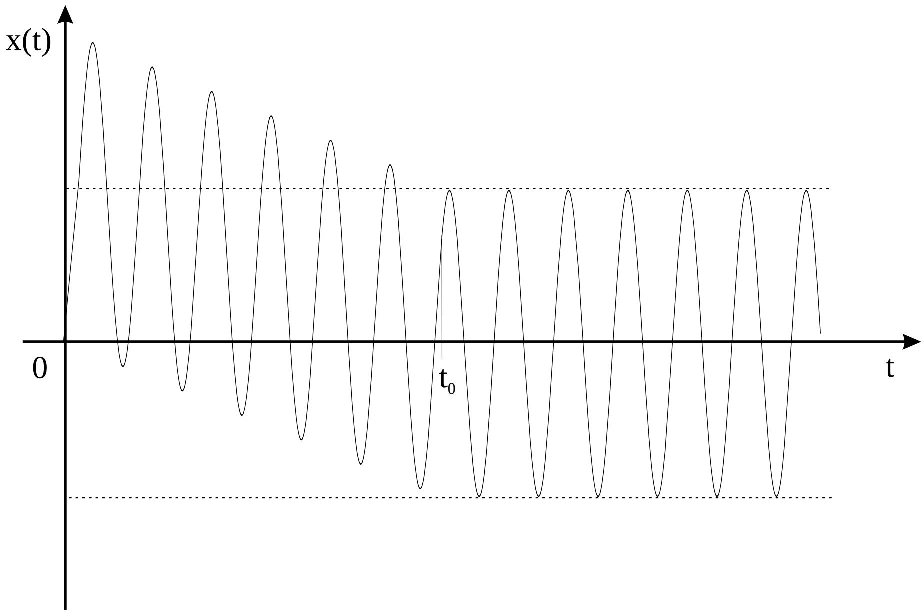
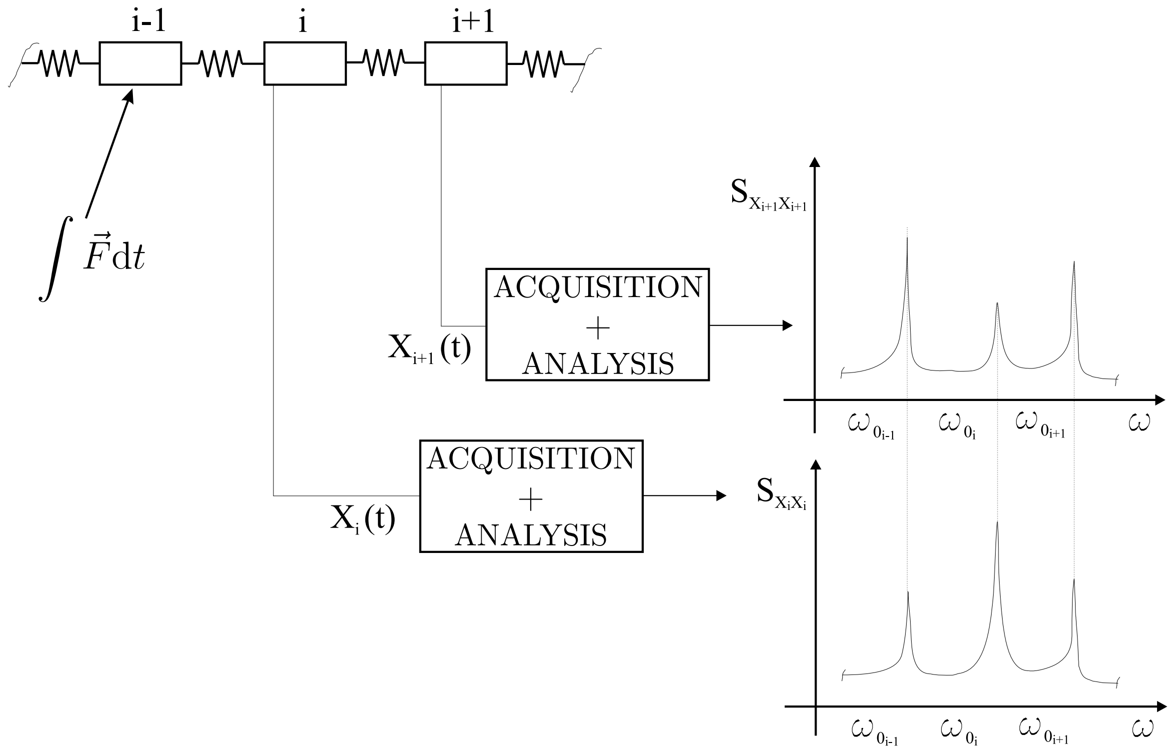


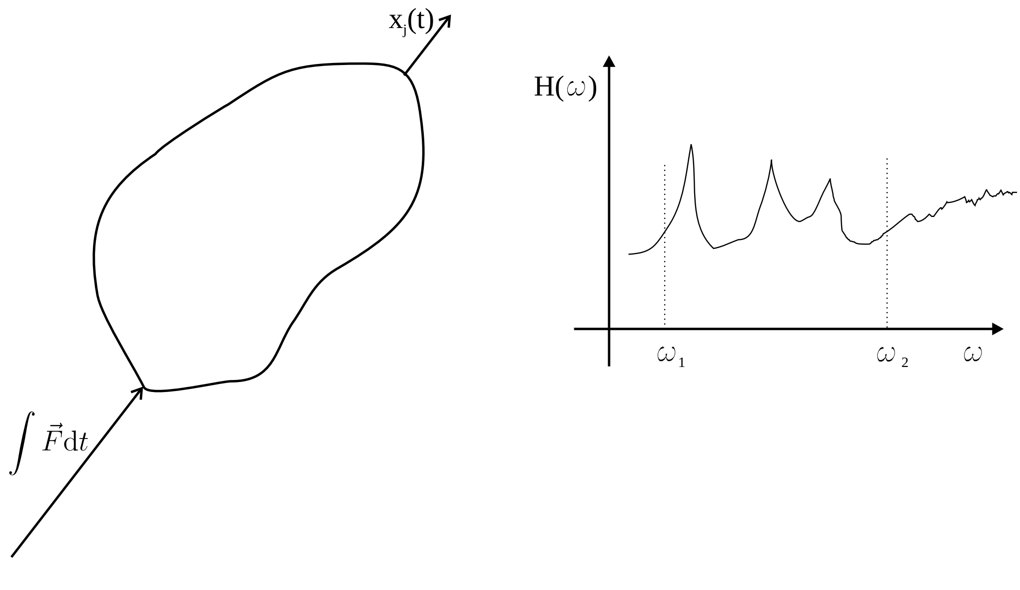
Figure 9: Response of the system in Fig. 7.

6 Use of impact excitation

6.1 Discrete systems



6.2 Continuous system - experimental structural dynamics



7 Experimental evaluation of model quality for calculating the bending oscillations in rotor dynamics

Figure 10: Shaft discretization.

	f_{01} [Hz]	Δ [%]	f_{02} [Hz]	Δ [%]
Experim. value	480.0		1050.0	
MPM	465.7	2.98	1044.4	0.56

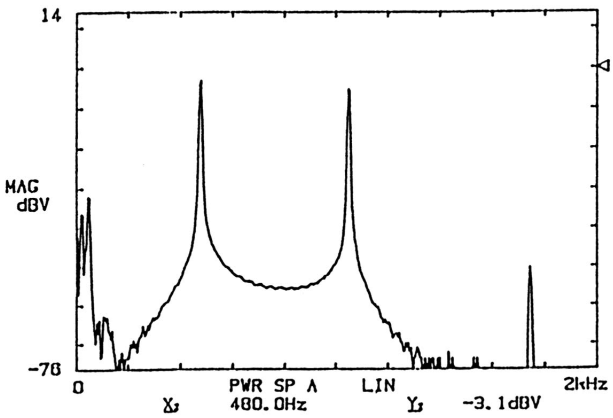
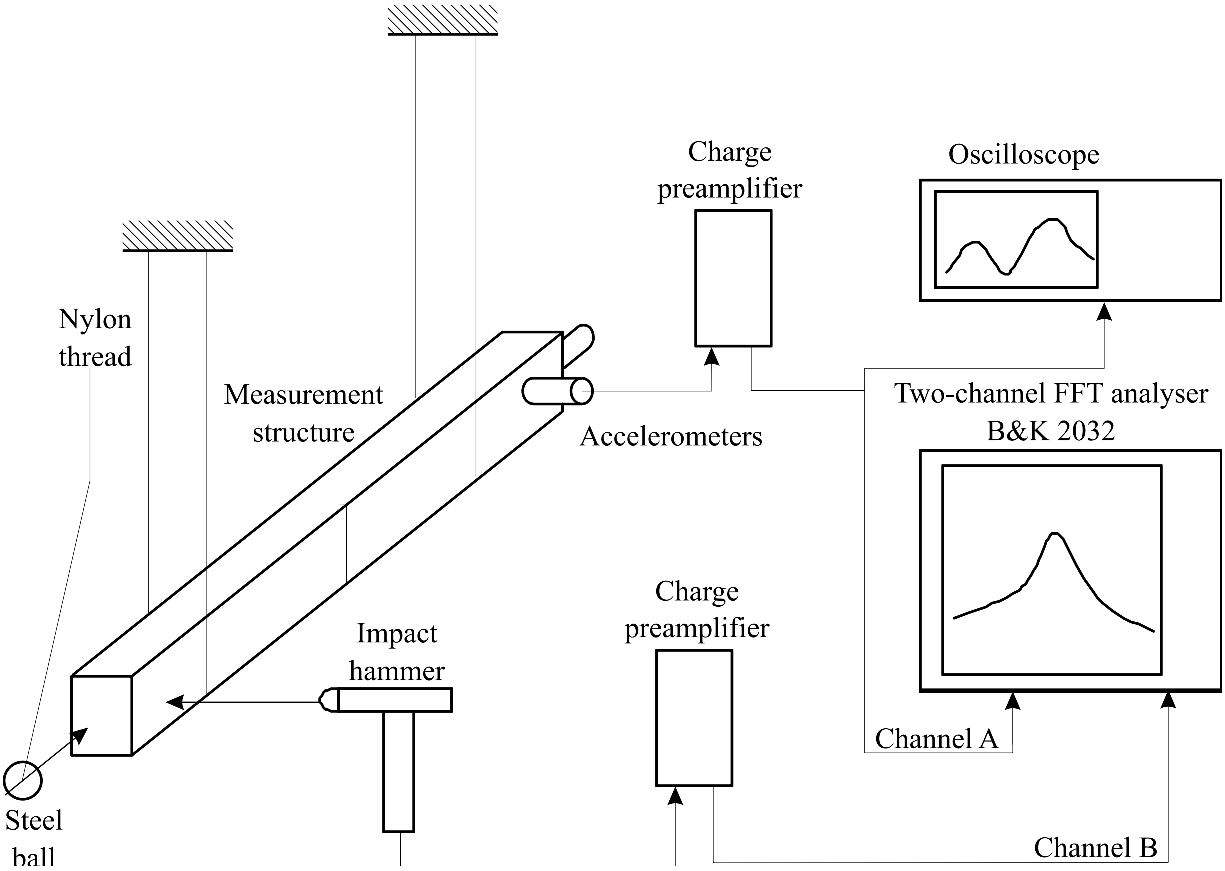
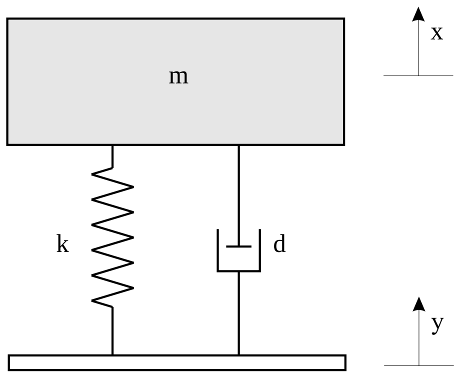


Figure 11: Spectral density diagram.

8 Non-destructive testing of damaged structures



9 Response to base excitation



Equation of motion:

$$m\ddot{x} = -k(x-y) - d(\dot{x} - \dot{y}) \tag{14}$$

By introducing a new variable $z = x - y$, the equation of motion (14) can be written in the form:

$$m\ddot{z} + d\dot{z} + kz = -m\ddot{y} \tag{15}$$

Notice the similarity of equations (15) and (16):

$$m\ddot{z} + d\dot{z} + kz = F \tag{16}$$

For a viscous-underdamped system, the impulse response function is written as:

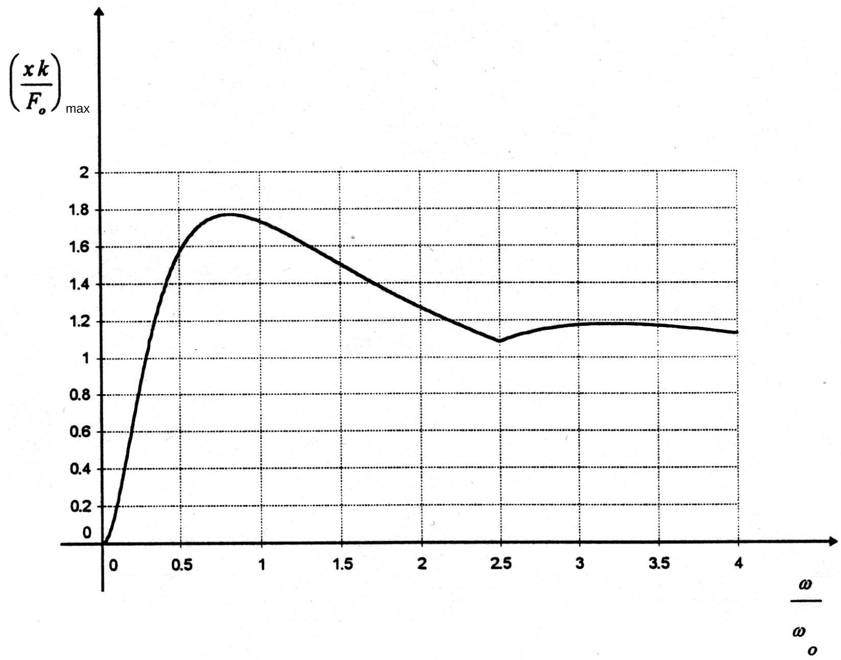
$$g(t) = \frac{1}{m\omega_{0d}} e^{-\delta\omega_0 t} \sin(\omega_{0d} t) \tag{17}$$

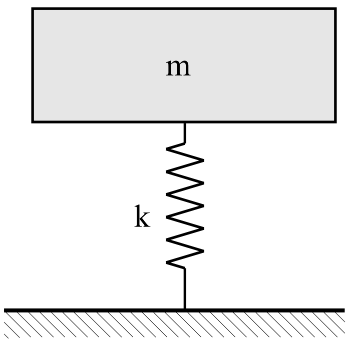
The solution of the equation of motion (14) is calculated by:

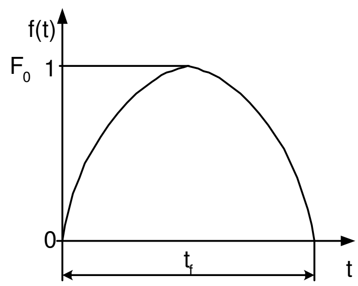
$$x(t) = \int_0^t f(t_1) g(t - t_1) \mathrm{d}t_1 \tag{18}$$

$$z(t) = -\frac{1}{\omega_{0d}} \int_0^t \ddot{y}(t_1) e^{-\delta\omega_0 (t-t_1)} \sin(\omega_{0d}(t - t_1)) \mathrm{d}t_1 \tag{19}$$

10 Impulse spectrum







$$\omega_0 = \sqrt{\frac{k}{m}} \quad \omega = \frac{\pi}{t_f}$$