1 Analytical statics

1.1 Tutorial objective

Using the principles of the Lagrange mechanics often proves to be useful for solving real-world problems. One of the areas which are typically subject to the use of these principles is determining the equilibrium states of multiple-degree-of-freedom systems, which are studied in this course under analytical statics.

The objective of this tutorial is to apply the theoretical knowledge from analytical statics on a simplified arm model, shown in Fig. 1. The tutorial consists of a practical study of the mechanical arm system and the analysis of its physical model. Two separate load examples will be studied and a comparison of the real and calculated equilibrium state of the studied system will be made.

Figure 1: Simplified arm model.

1.2 Physical model of the studied system

The physical model of the arm is schematically shon in Fig. 2, which consists of two pinned connections, two rigid bodies and two connecting springs. The studied system has two degrees of freedom. A wise choice for the independent coordinates would be the angular displacement of the individual rigid bodies φ_1 and φ_2. The y-component of the center of mass of each rigid body can be written with respect to the chosen coordinates as[1]:

$$
\begin{aligned}
y_{\mathrm{T}_1} &= -r_1\,\cos\varphi_1\,, &(1)\\
y_{\mathrm{T}_2} &= -L_1\,\cos\varphi_1 - r_2\,\sin\varphi_2\,. &(2)
\end{aligned}
$$

The spring extension must also be written with respect to the chosen coordinates. The distance between the axes of the spring mounting c is determined using the law of cosines, while the extension Δc is calculated by subtracting the initial length of the spring c_0:

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2\,a\,b\,\cos\left(90^\circ - \varphi_1 - \alpha - \beta + \varphi_2\right), &(3)\\
\Delta c &= c - c_0\,. &(4)
\end{aligned}
$$

The potential energy of the system is therefore calculated as:

$$
\mathrm{E}_p = m_1\,g\,y_{\mathrm{T}_1} + m_2\,g\,y_{\mathrm{T}_2} + \frac{k\,\Delta c^2}{2}\,. \tag{5}
$$

Since the studied system is conservative, the state of equilibrium can be found by the following stability criterion:

$$
-\frac{\partial \mathrm{E}_p}{\partial q_j} = 0\,;\ j = 1, 2\,. \tag{6}
$$

[1] The physical model enables a simplified study of the position of the center of mass of each rigid body, where the point T_1 is considered to lay on the line which connects the connecting pins, while T_2 lays on the line, connecting the axis of the connecting pin and the cylindrical spring in the hand of the model.

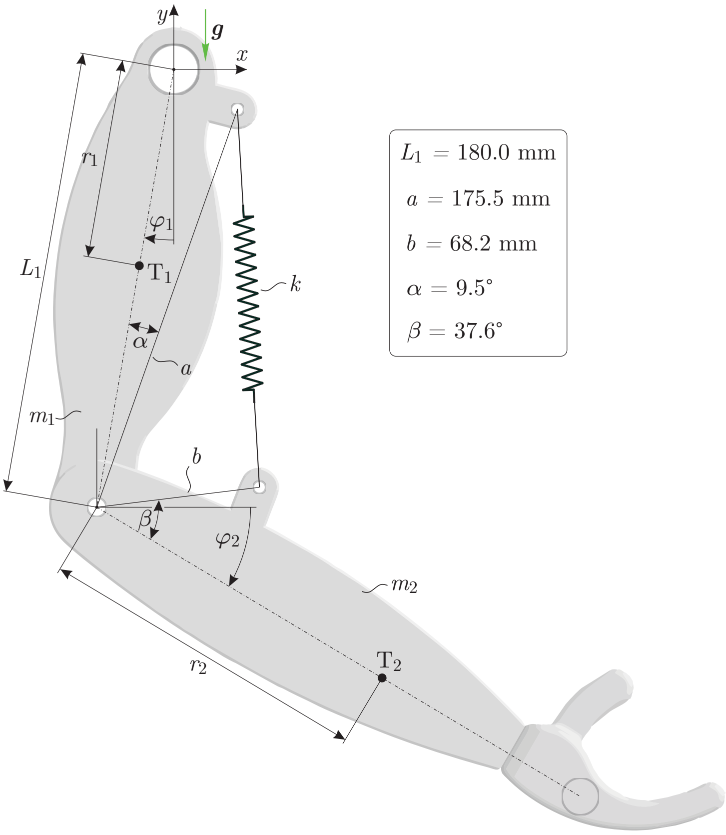

Figure 2: Physical model.

1.3 Instructions for performing the tutorial

1. Determining the unknown parameters of the physical model.

   a) Disassemble the studied model, such that the separate components will be available (upper arm, forearm and three springs connected in series).

   b) Measure the length of the springs connected in series in the undeformed configuration c_0 and determine the total stiffness of the springs k using weights.

   c) Using the scale, measure the mass of the individual components m_1 and m_2.

   d) For each system component (upper arm, forearm) verify the simplified studying of the center of mass, as described in the footnote 1. Using the technique of hanging, estimate the parameters r_1 and r_2.

2. Studying the physical model.

   a) Derive the equilibrium equations based on the equation for the system's potential energy and the determined parameters of the physical model.

   b) Choose an appropriate numerical method to solve the derived system of equations and calculate the values of φ_1 and φ_2 in the equilibrium state of the system.

3. Experiment - first load case

   a) Assemble the arm model and release it into the equilibrium state. In the first case, we simulate the arm in a relaxed state without any additional loads.

   b) Take a photo of the equilibrium state and by using a computer program (e.g. https://ij.imjoy.io/) estimate the values of φ_1 and φ_2 in the equilibrium state. Compare the measured values with the values calculate based on the numerical model.

4. Experiment - second load case

   a) In the second load case, the arm is studied with an additional load. The flexed muscle will be simulated by increasing the spring stiffness (by removing one of the springs connected in series) and a weight is added to the hand. Take a photo of the equilibrium state and estimate the values of φ_1 and φ_2 using an image processing program.

5. Adjusting the numerical model.

   a) According to the changes of the real model, update the numerical model and calculate the values of φ_1 and φ_2 in equilibrium for the second load case. Compare the numerically and experimentally obtained results. Use Fig. 3 as a reference.

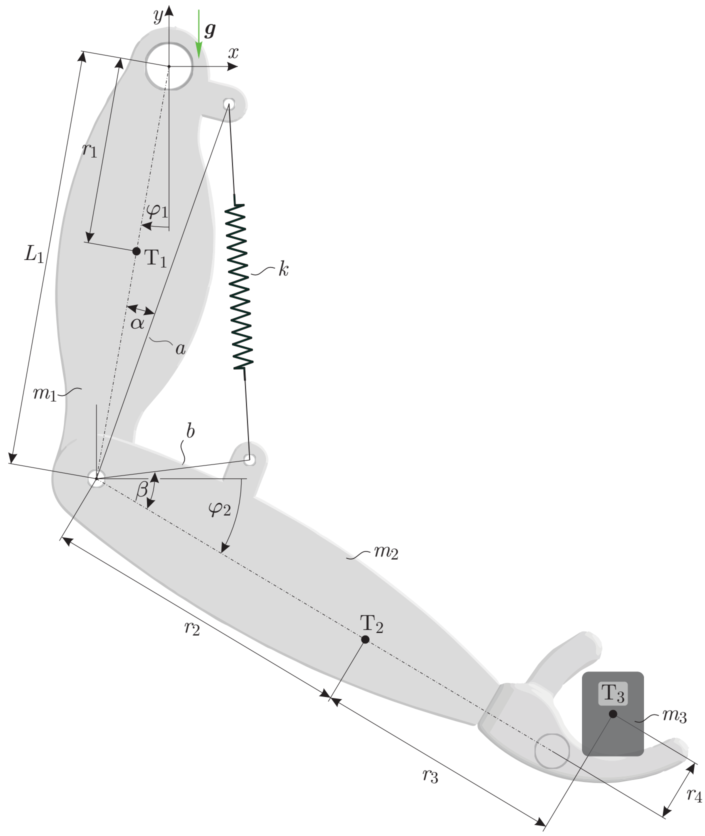

Figure 3: Physical model - load case 2.

1.4 Results

Write the obtained results into the Table below:

	Load case 1		Load case 2	
	Calculated value	Experimental value	Calculated value	Experimental value
$\varphi_1[^\circ]$				
$\varphi_2[^\circ]$				

2 Analytical dynamics

2.1 Tutorial objective

Spontaneous synchronization is an interesting natural phenomenon, which occurs in a population of oscillating units which are coupled by a certain interaction mechanism. Christiaan Huygens described the phenomenon in the 17th century by observing two mechanical clocks, which is used in the modern day in many different engineering applications.

The objective of the tutorial is to present the concept of spontaneous synchronization and connect it with the obtained knowledge from analytical dynamics. The goal is to establish a physical model, which can describe the phenomenon of synchronizing two metronomes on a common sliding base. The studied system is shown in Fig. 4.

Figure 4: Metronomes on a common sliding base.

2.2 Physical model of the studied system

The physical model of the studied system is shown in Fig. 5. It consists of two metronomes and a connecting cart, which can slide along the surface. The studied system is a three degrees of freedom system, where a wise choice for the independent coordinates are the angular displacements of the pendulums on the metronomes φ_1 and φ_2, and the displacement of the sliding base x_3. With respect to the chosen independent coordinates, the distance between the mounting and the pendulum center of mass r can be written as:

$$r = \frac{-m_1\, l_1 + m_2\, l_2}{m_1 + m_2}\,. \tag{7}$$

When calculating the mass moment of inertia, the mass of the bar will be neglected, while the weights will be considered as mass points. The center of mass moment of inertia is written as:

$$J_{\mathrm{T}} = m_1\, (l_1 + r)^2 + m_2\, (l_2 - r)^2\,. \tag{8}$$

The coordinates of the center of mass for each pendulum can be written as a function of the chosen independent coordinates[2]:

$$x_{\mathrm{T}_1} = x_3 - r\,\sin\varphi_1 + C_1\,, \tag{9}$$
$$y_{\mathrm{T}_1} = r\,\cos\varphi_1\,, \tag{10}$$
$$x_{\mathrm{T}_2} = x_3 - r\,\sin\varphi_2 + C_2\,, \tag{11}$$
$$y_{\mathrm{T}_2} = r\,\cos\varphi_2\,. \tag{12}$$

[2] C_1 and C_2 denote arbitrary constants, describing the shift between the metronomes in the horizontal direction and do not affect the equations of motion

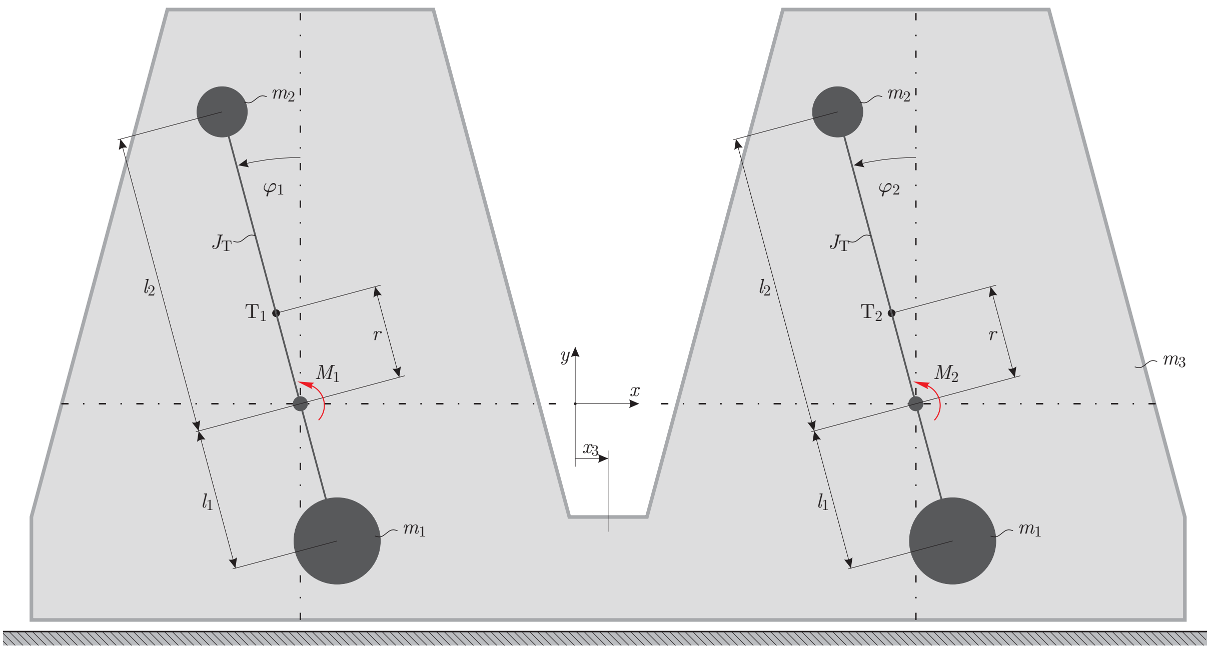

Figure 5: Physical model.

The potential and kinetic energy of the system can be written as:

$$\mathrm{E}_p = (m_1 + m_2)\, g\, y_{\mathrm{T}_1} + (m_1 + m_2)\, g\, y_{\mathrm{T}_2}\,, \tag{13}$$

$$\mathrm{E}_k = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2_{\mathrm{T}_1} + \frac{1}{2}(m_1 + m_2)\,\dot{y}^2_{\mathrm{T}_1} + \frac{1}{2}(m_1 + m_2)\,\dot{x}^2_{\mathrm{T}_2} + \frac{1}{2}(m_1 + m_2)\,\dot{y}^2_{\mathrm{T}_2} + \\ + \frac{1}{2} J_\mathrm{T}\,\dot{\varphi}_1{}^2 + \frac{1}{2} J_\mathrm{T}\,\dot{\varphi}_2{}^2 + \frac{1}{2}(m_3)\,\dot{x}_3^2\,. \tag{14}$$

The effect of the driving mechanism of the metronome to the individual pendulum will be considered as a moment M_i, which will be described in a simplified way using the van der Pol's term as:

$$M_i = \epsilon \left(\left(\frac{\varphi_i}{\phi_0} \right)^2 - 1 \right) \dot{\varphi}_i\,;\; i = 1, 2\,, \tag{15}$$

where this term increases the angular velocity when $\varphi_i < \phi_0$, and decreases the angular velocity when $\varphi_i > \phi_0$. For small values of ϵ, this term will produce a steady oscillation with the amplitude of approximately $2\,\phi_0$. The parameter ϵ determines the duration of the time it takes to achieve a stable state. Considering the real system, an appropriate value of the parameter would be $\epsilon = 0.0001 \frac{\mathrm{kg\,m^2}}{\mathrm{s}}$.

Three equations of motion can be derived for the studied three degree of freedom system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_1} - \frac{\partial L}{\partial \varphi_1} = -M_1\,, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\varphi}_2} - \frac{\partial L}{\partial \varphi_2} = -M_2\,, \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_3} - \frac{\partial L}{\partial x_3} = 0\,, \tag{18}$$

where L denotes the Lagrangian, which is written as a difference of the kinetic and potential energy $L = \mathrm{E}_k - \mathrm{E}_p$.

It must be emphasized that the spontaneous synchronization can occur only if there is sufficient interactions between the system components. Therefore, the influence of the pendulum dynamics must sufficiently excite the oscillation of the common base, which is accomplished in the current example by increasing the oscillation frequency, decreasing the mass of the other components and reducing friction.

2.3 Instructions for performing the tutorial

1. Analysis of the metronome operation

    a) Analyze the steady response of the metronome pendulum for different initial conditions and compare the observations with a physical explanation of van der Pol's term for describing the influence of the driving mechanism of the metronome.

    b) Estimate the value of ϕ_0 based on the observation (should be approximately half of the steady metronome oscillation amplitude).

2. Estimation of the physical model's parameters.

    a) Due to the complexity, the metronome will not be disassembled. Depending on availability, estimate or measure the values of l_1, l_2, m_1 and m_2.

    b) Determine the value of m_3 using a scale and the estimated values m_1 and m_2.

3. Studying the physical model.

    a) Derive the equations of motion from the equations for the potential and kinetic energy of the system.

    b) Solve the derived system of differential equations using an appropriate numerical method. Plot the angular displacement of the pendulums φ_1 and φ_2 with respect to time and look out for spontaneous synchronization.

4. Performing the experiment

    a) Perform the experiment for four distinct operating cases:

        i Enabled sliding of the cart, low oscillation frequency and an arbitrary initial condition.

        ii Enabled sliding of the cart, high oscillation frequency and a similar initial condition for both pendulums.

        iii Enabled sliding of the cart, high oscillation frequency and the initial conditions of the pendulums in the antiphase.

        iv Disabled sliding of the cart, high oscillation frequency and an arbitrary initial condition.

    For each of these cases observe the system behavior and write down whether the synchronization occurred. If it did, also write down the time to synchronization and any additional observations.

5. Simulation using the numerical model:

    a) Try to simulate a similar behavior as observed in all four operating cases by appropriately setting the parameters of the numerical model:

        i Set the parameters m_1, m_2, l_1 and l_2 such that the oscillation frequency is low and observe the time it takes to achieve synchronization.

        ii Adjust the parameters m_1, m_2, l_1 and l_2 such that the oscillation frequency is significantly higher than in the previous case and observe the time it takes to achieve synchronization.

        iii Take an additional interest in the synchronization of pendulums starting in the antiphase by separately analyzing an ideal case and the case with a slight numerical difference between the initial conditions of the individual pendulums. Consider whether or not the aniphase synchronization is a stable or unstable state (in lightly damped systems).

        iv Since an undamped and a completely freely mounted physical model is considered, the interaction between the components can be most easily decreased by significantly increasing the value of m_3. Confirm whether in this case the spontaneous synchronization also occurs (in a reasonable time).

2.4 Results

Experimental observations for different operating cases:

	Synchronization - YES/NO	Time to synchronization [s]	Additional observations or notes
i			
ii			
iii			
iv			