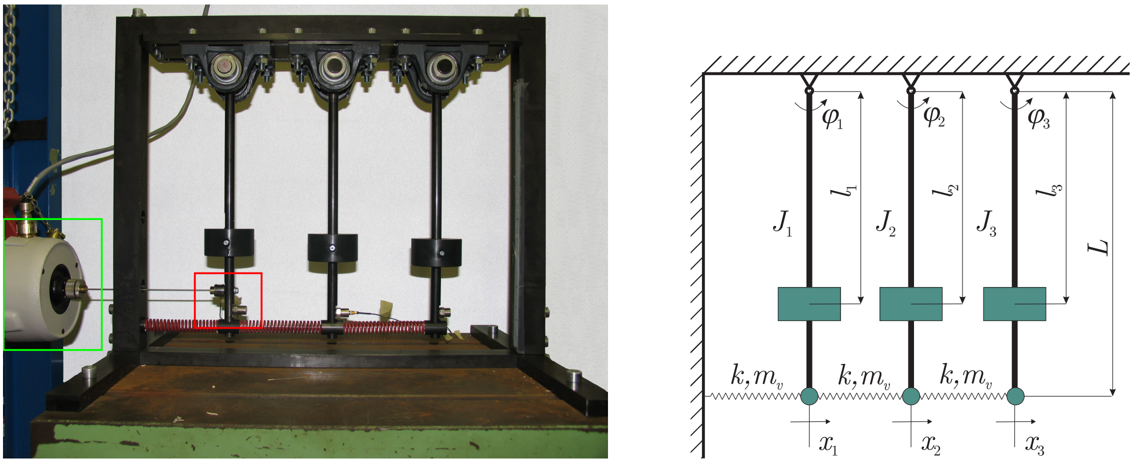
3 Characterization of multiple-degree-of-freedom system

3.1 Tutorial purpose

Real systems can rarely be described as single-degree-of-freedom systems, which should be taken into account when characterizing systems.
The purpose of this tutorial is to understand multiple-degree-of-freedom systems and experimentally determine eigenfrequencies and eigenvectors form the frequency response functions.

3.2 Task definition

Calculate the eigenfrequencies of the system in Fig. 1. Measure the eigenfrequencies of the system. Consider the infinitesimal strain theory. Damping can be neglected.



((a)) ((b))

Figure 1: Multiple-degree-of-freedom system: (a) real system; (b) mathematical model.

3.3 Mathematical model

The system in Fig. 1(a) is simplified as shown in Fig. 1(b). The equations of motion can be deduced using Newton's second law of motion or by using the second form of Lagrange equations. For improved readability, the latter will be used. It is assumed that $x_1 > x_2 > x_3$. The Lagrangian $\mathfrak{L}$ is defined as:

$$\mathfrak{L} = E_k - E_p, \tag{1}$$

where E_k is the kinetic energy:

$$E_k = \frac{1}{2}J_1\dot{\varphi_1}^2 + \frac{1}{2}J_2\dot{\varphi_2}^2 + \frac{1}{2}J_3\dot{\varphi_3}^2 + \frac{m_v}{6}\cdot\left[\dot{x}_1^2 + (\dot{x_1} - \dot{x_2})^2 + (\dot{x_2} - \dot{x_3})^2\right], \tag{2}$$

and E_p denotes the potential energy of the system:

$$E_p = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}k(x_2 - x_3)^2. \tag{3}$$

J_i denotes the mass moment of inertia about the axis of rotation of the ith bar with a weight, while m_v denotes the spring mass. The mass moment of inertia of the individual bar is calculated as:

$$J_i = J_p + J_{ui} + J_n = \frac{1}{3} m\, L^2 + M\, l_i^2 + \frac{M}{12}\left(3(R^2 - r^2) + H^2\right) + m_n L^2, \tag{4}$$

where J_p is the mass moment of inertia about the axis of rotation, J_{ui} is the mass moment of inertia of the weight about the axis of rotation, and J_n is the mass moment of inertia of the spring attachment about the axis of rotation. M denotes the mass of the weight, m the bar mass and m_n the mass of the spring attachment. R is the outer radius of the weight, r is the hole radius (and also the bar radius). L is the bar length and l_i is the distance of the weight from the axis of rotation. H denotes the height of the weight.

The equations of motion are deduced by calculating the derivative of the Lagrangian with respect to each coordinate; the equality $x_i = L \sin(\varphi_i)$ is also taken into account.

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\varphi}_i}\right) - \frac{\partial \mathfrak{L}}{\partial \varphi_i} = 0 \tag{5}$$

The derivatives of equations (2) and (3) have to be linearized (by considering $\sin(\varphi_i) \approx \varphi_i$ and $\cos(\varphi_i) \approx 1$). The response is calculated by assuming a harmonic response of the system and using the substitution $\varphi_i(t) = \phi_i \sin(\omega t)$. Based on the above equations, the mass matrix is written as:

$$\mathbf{M} = \begin{bmatrix} J_1 + \frac{2}{3} L^2\, m_v & -\frac{1}{3} L^2\, m_v & 0 \\ -\frac{1}{3} L^2\, m_v & J_2 + \frac{2}{3} L^2\, m_v & -\frac{1}{3} L^2\, m_v \\ 0 & -\frac{1}{3} L^2\, m_v & J_3 + \frac{2}{3} L^2\, m_v \end{bmatrix} \tag{6}$$

while the stiffness matrix is:

$$\mathbf{K} = \begin{bmatrix} 2k\, L^2 & -k\, L^2 & 0 \\ -k\, L^2 & 2k\, L^2 & -k\, L^2 \\ 0 & -k\, L^2 & 2k\, L^2 \end{bmatrix}. \tag{7}$$

The system of the equations of motion can be therefore written as:

$$\left(-\omega^2 \cdot \mathbf{M} + \mathbf{K}\right) \cdot \boldsymbol{\phi} = \mathbf{0}, \tag{8}$$

where $\boldsymbol{\phi}$ denotes the amplitude vector and $\mathbf{0}$ denotes the zero vector. The eigenfrequencies of the system are found by solving:

$$\det\left[-\omega^2 \mathbf{M} + \mathbf{K}\right] = 0. \tag{9}$$

The eigenvectors of the system are found by inserting the individual eigenfrequency into Eq. (8). This yields a system of two independent and one dependent equation, the eigenvectors are normalized.

3.4 Experiment

3.4.1 Frequency response function

Frequency response function (FRF) determines the modal parameters of the system (eigenfrequencies, eigenvectors, damping). Generally, a frequency response function represents a ratio between the input and output signal with respect to frequency. The frequency response function is written as:

$$H(\omega) = \frac{O(\omega)}{I(\omega)}, \tag{10}$$

graphically shown in Fig. 2.

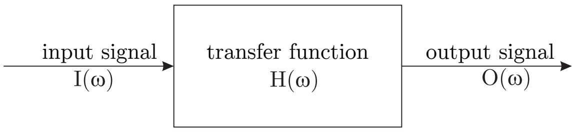


Figure 2: Frequency response function scheme.

A frequency response function is determined by the amplitude and phase, as shown in Fig. 3. Based on the chosen (measured) input and output quantities, different frequency response functions can be defined. In this tutorial, a frequency response function will represent the ratio between the acceleration and force. The eigenfrequencies and eigenvectors can be determined directly from the frequency response functions.

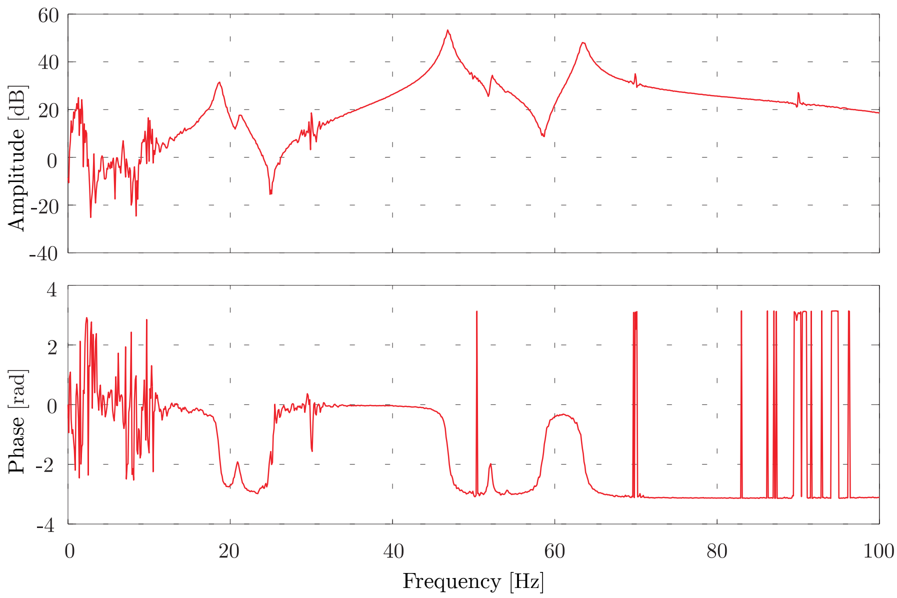


Figure 3: Amplitude (above) and phase plot (below) of the FRF.

3.4.2 Measuring resonance frequencies

In the experiment, the system is excited with broadband random excitation using a shaker. This type of excitations ensures the power of the excitation signal is uniformly distributed throughout all frequencies. The shaker is shown in a red box in Fig. 1(a). The input signal for the shaker is generated using LabVIEW.

The objective of the measurements is determining the system response. The excitation force is measured at the end of the shaker bar using a load cell, while the accelerations are measured on each bar using an accelerometer. The load cell and accelerometer on the first bar are shown in a green box in Fig. 1(a).

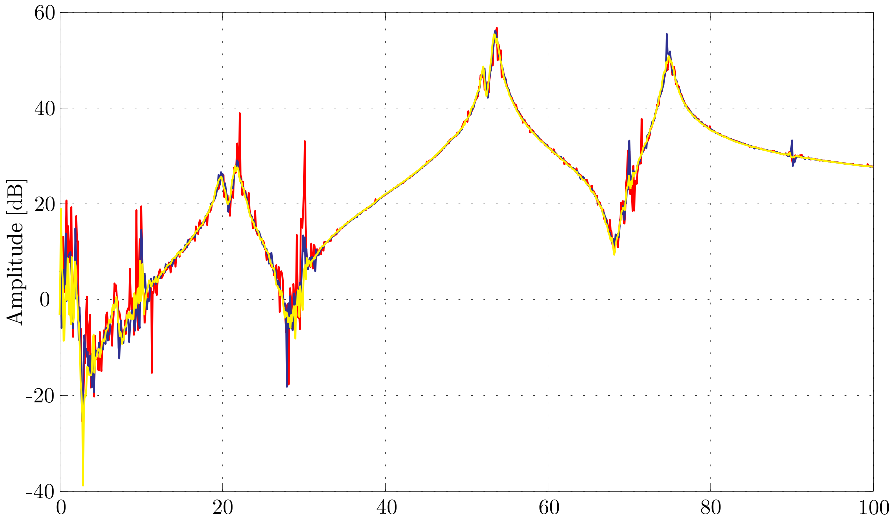


Figure 4: The effect of averaging. Legend: (▬) no averaging, (▬) two averaged signals, (▬) ten averaged signals.

The sensor data were captured using an analog to digital converter and were subsequently processed using the program LabVIEW. The amplitude and phase plot of the frequency response function (acceleration/force). The filters, windowing and averaging method must be set appropriately for the data processing.

3.4.3 Influencing factors

Filtering effect Filters are used when we wish to remove or hide some frequencies present in the signal. A wide range of filters exists, the most typical being low-pass, high-pass, band-pass and band-stop filter.

Averaging effect If the signal contains noise, the majority of the noise can be removed (smoothed) by averaging. Depending on where the noise enters the system (input, output or both), different types of averaging can be used. An example of averaging is shown in Fig. 4.

Windowing effect Window functions are used to decrease the boundary effects (consequence of noise in Fourier transform) and to remove the discontinuities of the signal when processing real data (the repeatability of real signals is lower than assumed). Windowing is used prior to the Fourier transform of a signal to reduce *leakage*.
A window is defined as a function in some interval. This function is nonzero inside this interval and equal to zero outside the interval. Multiplying the function which defines the signal with the function defining the window yields a nonzero result if the window function is nonzero. Although using a window changes the signal, an appropriate use of the window can reduce the discontinuities at the borders of the window. Numerous types of windows exist. Some are used more generally, while others are used in very specific applications. The simplest type is a rectangular window (usually this is regarded as a signal without any window). This window produces large side lobes, which is why it is not recommended, except for short signals. The most widely used is the Hanning window. Hamming window is similar, except its value is not zero at the endpoints. It is used mainly for signals, where the frequencies are closely spaced.

3.5 Course of the tutorial

The subject of the analysis is a multiple degree of freedom system, shown in Fig. 1(a). The properties of the system are are: $m = 0.536$ kg, $M = 1.513$ kg, $m_v = 0.105$ kg, $m_n = 0.052$ kg, $R = 0.0325$ m, $r = 0.0075$ m, $H = 0.04$ m, $k = 25.734$ N/mm, $L = 0.38$ m. The expected results of the tutorial are:

1. for specific values of l_1, l_2 and l_3 calculate the eigenfrequencies and eigenvalues of the mathematical model of the system;
2. set the weights to the set values of l_1, l_2 and l_3 and measure the eigenfrequencies;
3. calculate the error between the calculated and measured eigenfrequencies.

The calculation and measurements must be executed for two different sets of data.

3.6 Theoretical questions

1. How are the eigenfrequencies determined from the frequency response function? What about eigenvectors? Are the eigenvectors normalized in this case as well? Do the eigenvectors and eigenfrequencies depend on the frequency response functions?
2. What is white noise? Which other types of excitation can be used?
3. Why can we neglect the effect of damping in the current example? Can this approach be used generally?
4. Which types of window functions do we know of? Why is there a need for the use of windows?
5. Which types of filters do we know of? Why is there a need for the use of filters? What is the difference between filters and windows?
6. Deduce the equations of motion for an undamped three-degree-of-freedom system.
7. What is the effect of averaging signals and why is averaging used?
8. Why do we mound a load cell between the shaker and the system? How does a load cell work?

3.7 Review of measurements and results

Symbol	Value	Unit
1. distribution of bars		
l_1		mm
l_2		mm
l_3		mm
Analytical calculation		
$f_{0,1}$		Hz
$f_{0,2}$		Hz
$f_{0,3}$		Hz
Measurements		
$f_{0,1}$		Hz
$f_{0,2}$		Hz
$f_{0,3}$		Hz
2. distribution of bars		
l_1		mm
l_2		mm
l_3		mm
Analytical calculation		
$f_{0,1}$		Hz
$f_{0,2}$		Hz
$f_{0,3}$		Hz
Measurements		
$f_{0,1}$		Hz
$f_{0,2}$		Hz
$f_{0,3}$		Hz