5 Oscillation of continuous systems

5.1 Tutorial objective

According to the fundamental principles of continuum mechanics, the mass and stiffness properties of a continuous system are continuously distributed throughout the system. In this course, only some basic construction elements were considered when studying continuous systems (string, bar, shaft and beam), assuming linearity, infinitesimal strain theory, absence of damping and uniaxial stress.

The objective of the tutorial is the application of the theoretical knowledge about the dynamics of continuous systems to the analysis of belt drives. The experimental assembly is shown in Fig. 1. The tutorial consists of experimental-analytical consideration of the belt preload and the analysis of its dynamic response in different operational conditions.

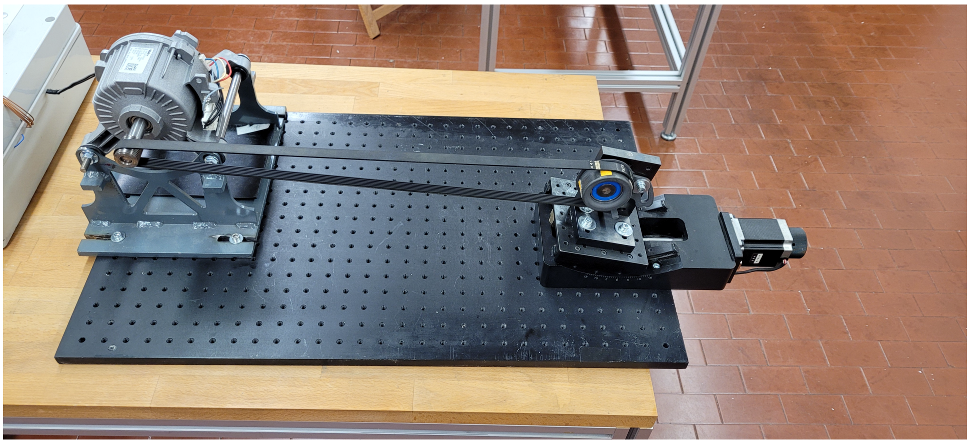

Figure 1: Belt drive system.

5.2 Physical model of the system

A simplified physical model of the belt will be considered for the calculation, where each side of the belt will be considered as a string with mass per lenght μ and free length L. A uniform preload force $\boldsymbol{N}$ will be assumed, as shown in Fig. 2. The free transversal oscillation of a string is described by the second order differential equation:

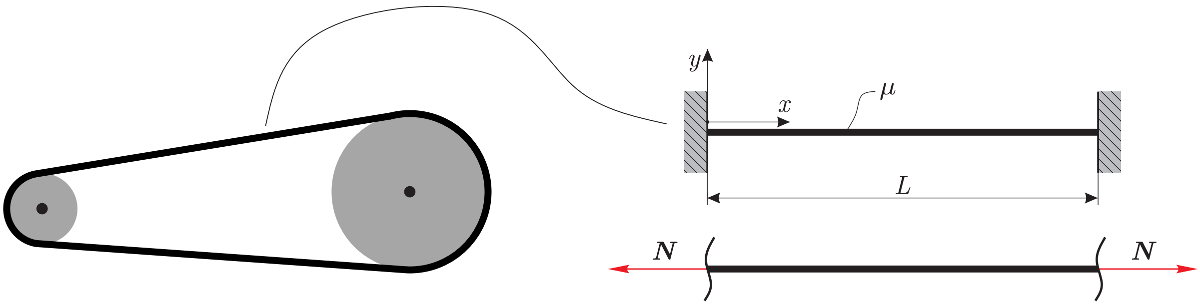

Figure 2: Physical model.

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c}\frac{\partial^2 y(x,t)}{\partial t^2}, \tag{1}$$

where the parameter c denotes the wave speed:

$$c^2 = \frac{N}{\mu}. \tag{2}$$

If the unknown function $y(x,t)$ can be written as a product of spatial and temporal functions $y(x,t) = Y(x)\,T(t)$, the spatial part of the differential equation can be written as:

$$Y''(x) + \left(\frac{\omega}{c}\right)^2 Y(x) = 0, \tag{3}$$

where ω denotes the angular frequency. The solution to the above equation is written below:

$$Y(x) = C\,\cos\left(\frac{\omega}{c}x\right) + D\,\sin\left(\frac{\omega}{c}x\right). \tag{4}$$

The constants C and D depend on the boundary conditions, which can be calculated for the given cantilever (or pinned) support:

$$\begin{aligned} Y(x=0)=0 &= C\,\cos\left(\frac{\omega}{c}0\right) + D\,\sin\left(\frac{\omega}{c}0\right), \\ 0 &= C, \end{aligned} \tag{5}$$

$$\begin{aligned} Y(x=L)=0 &= D\,\sin\left(\frac{\omega}{c}L\right), \\ 0 &= \sin\left(\frac{\omega}{c}L\right). \end{aligned} \tag{6}$$

The solution for the nth angular eigenfrequency can be written from the above conditions as:

$$\omega_n = n\frac{\pi\,c}{L} = n\frac{\pi}{L}\sqrt{\frac{N}{\mu}}; \quad n \in \mathbb{N}, \tag{7}$$

while the corresponding nth mode shape is calculated as:

$$Y(x) = D\sin\left(\frac{\omega_n}{c}x\right) = D\sin\left(n\frac{\pi}{L}x\right); \quad n \in \mathbb{N}. \tag{8}$$

The preload force (i.e. the axial force in the belt) can be easily calculated from the experimentally determined (first) eigenfrequency, assuming the material properties are known:

$$N = \frac{L^2\,\omega_n^2\,\mu}{n^2\,\pi^2} = \frac{L^2\,\omega_1^2\,\mu}{\pi^2}. \tag{9}$$

The derivation of the physical model assumes a stationary string. In real applications, belts typically move with a known speed, which affects the dynamic response of the belt in reality. The angular eigenfrequencies of the belt, considering a constant speed v, can be calculated by the following equation:

$$\omega_n = n\,\frac{\pi}{L}\,\frac{N-\mu\,v^2}{\sqrt{N\,\mu}}; \quad n \in \mathbb{N}. \tag{10}$$

Apart from the simple string-based physical model, more advanced models are used in practice, where the belt is considered as a superposition of a string and a (pin-supported) beam, which allows to take into account the bending stiffness. In such case, the angular eigenfrequency is calculated as:

$$\tilde{\omega}_n^2 = \omega_n^2 + \omega_{n,\mathrm{nos}}^2, \tag{11}$$

where $\omega_{n,\mathrm{nos}} = \left(\frac{n\,\pi}{L}\right)^2\left(\frac{E\,I}{\mu}\right)^{1/2}$.

5.3 Instructions for performing the tutorial

1. Determining the unknown parameters of the physical model:
   a) Determine the free length of a single side of the belt L and mass per unit length μ.
   b) Estimate the elastic modulus of the belt E and the second moment of area of the belt cross section.
2. Free belt oscillation:
   a) Assemble the experimental assembly and preload the belt using the belt tensioner.
   b) Excite the free oscillation of the belt, estimate the first transversal eigenfrequency and estimate the preload force.
   c) Instead of the sound analyzer use a smartphone app for the spectral analysis (e.g. Phyphox or Spectroid) and compare the results with the sound analyzer.
   d) Achive the preload force of 500 N by adjusting the belt tensioner.
3. Forced belt oscillation:
   a) Think about what are the sources of the excitation in belt drives and estimate the potentially problematic intervals of the operational speed for this experimental assembly.
   b) Start the belt drive and observe the belt drive dynamics at different operating speeds and different acceleration regimes
   c) For a better visualization use a strobe light or a smartphone app (e.g. Strobily or Strobe Light).