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Lagrange - deduction of equality

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During the derivation of the Lagrange equations, two equalities were considered, which will be proven in the following.

1 Proof of the first equality

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}. \tag{1}$$

The derivation starts by writing the position vector as a function of the generalized coordinates

$$\mathbf{r}_i = \mathbf{r}_i(q_j), \quad j = 1, \ldots, N. \tag{2}$$

The generalized coordinates are indeed a function of time; $q_j = q_j(t)$. The derivative of the position vector with respect to time is calculated[1]

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_i) = \dot{\mathbf{r}}_i = \sum_{j=1}^{N} \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j. \tag{3}$$

The derivatives of both sides of equation (3) with respect to the generalized velocities are calculated, leading to the relation

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}. \tag{4}$$

[1] The equation is valid for scleronomic systems, i.e. the position vector is not explicitly time-dependent.

2 Proof of the second equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}. \tag{5}$$

Similarly, the derivation starts with the position vector being time-dependent, as in equation (2). The velocity is obtained by calculating the derivative of (2) with respect to time (3)[2]

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial \mathbf{r}_i}{\partial q_N}\dot{q}_N + \frac{\partial \mathbf{r}_i}{\partial t}. \tag{6}$$

The derivative of equation (6) with respect to the jth generalized coordinate is calculated, yielding

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \frac{\partial^2 \mathbf{r}_i}{\partial q_1 \partial q_j}\dot{q}_1 + \frac{\partial^2 \mathbf{r}_i}{\partial q_2 \partial q_j}\dot{q}_2 + \ldots + \frac{\partial^2 \mathbf{r}_i}{\partial q_N \partial q_j}\dot{q}_N + \frac{\partial^2 \mathbf{r}_i}{\partial t \partial q_j}. \tag{7}$$

Now, the derivative with respect to time of the partial derivative of the position vector with respect to the jth generalized coordinate is calculated, leading to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_1}\dot{q}_1 + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_2}\dot{q}_2 + \ldots + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_N}\dot{q}_N + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t}. \tag{8}$$

Under the assumption that the position vector's secont partial derivatives are continuous, the order of the partial derivatives is interchangeable, therefore, the equations (7) and (8) are equivalent, proving the second equality, i.e. equation (5).

[2] This represent a generalization of equation (3): the position vector can be explicitly dependent on time.