Frequency-based structural modification for the case of base excitation

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Abstract

When estimating a structure's fatigue life during vibrational test the stress frequency-response function (SFRF) to the base excitation is required. The response to this base excitation can be numerically obtained by solving the equilibrium equations for each frequency of interest. In this research we propose a new method, that can be used to obtain the SFRF of a base-excited structure using the modal model of the unconstrained structure, only. By further developing the idea of a structural modification using the response function this research significantly reduces the computation time and the amount of data sent to the fatigue-analysis software. The new method is presented on two numerical examples: a simple beam structure and a Y-shaped structure. Using numerical examples, the effects of the modal truncation, the matrix singularity and the damping are discussed.

Keywords: Structural modification, Vibration testing, Stress response function, Modal model

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1. Introduction

The base excitation is considered when the excitation is prescribed in terms of the kinematic motion as an acceleration or displacement profile instead of the excitation force at the structure's drive points. The base excitation of the structure is typically performed with an electro-dynamic shaker and represents a standard practice in environmental and seismic testing, regularly used in automotive, aerospace, electronics and civil-engineering industry [1–3]. When performing environmental testing in order to verify structure's durability the damage to the structure is introduced by the structure's dynamic stress response [4] and can lead to fatigue failure. The time-to-failure can be estimated using various fatigue analyses of the stress frequency-response function (SFRF) [5, 6]. However, the structure's dynamic response to the base motion is different from the response to the force excitation. This research introduces a new method for obtaining the structure's response to base motion from the response to the force excitation of the unconstrained structure.

With numerical finite-element models it is possible to obtain the SFRF in the case of base excitation by solving the equilibrium matrix equation and then calculating the nodal stress from the nodal displacement solution for the nodes and the frequency range of interest [7]. As the size of the numerical model, the frequency range of interest and the frequency resolution increase the calculation time and the amount of data that needs to be collected and sent to the vibrational fatigue postprocessor increases as well. To reduce the calculation time and to avoid assembling a large amount of data only the modal model can be exported and the SFRF can be obtained using the mode-superposition method [8]. However, the mode-superposition method can be utilized for the force excitation only. Currently, for the case of base excitation the frequency response can only be obtained directly (by using mass, stiffness and damping matrices). This problem arises due to the different excitation forces, that are in the case of base excitation inertial. However, base excitation is very frequent type of loading in environmental and seismic testing. Consequently, much research in the field of the base excitation has been done with the applications to the experiment. Béliveau *et al.* [9] identified modal parameters from the base excitation of the dynamic system. Furthermore, by measuring the interface forces Füllekrug and Sinapius [3] obtained mass-normalised mode-shapes and generalized and effective masses of the dynamic system with the multi-axial base excitation. The problem of verifying the interface properties of the constrained system was studied by Blair [2] with the use of different free-free configurations as alternatives to base-excitation testing.

The theory of the dynamic modification and coupling of structures [8, 10] is a standard approach to predicting the changes in the response of modified or coupled structures from the response of the initial structure or the substructures being coupled. Generally, two approaches can be applied for a dynamic modification, *i.e.*, the modal approach and the impedance approach [11–13]. This research focuses on the impedance method, which was introduced by Crowley *et al.* [14] and further developed by Jetmundsen *et al.* [15] with a new structural modification using the response function (SMURF) definition, which requires the inversion of only one transmissibility matrix. The SMURF technique is convenient when there is no mass and stiffness information about the structure or its modification available. This was the case in the research of Massey *et al.* [16], where the response of the modified H-frame was predicted from the experimentally obtained frequency-response functions of the initial H-frame and the additional beam. Ren *et al.* [17] introduced a generalized frequency-substructuring definition to extend the applicability of the frequency-based modification. Recently, a reformulation of the frequency substructuring using the dual-domain decomposition method was presented by de Klerk *et al.* [18]. This formulation was adopted by D'Ambrogio *et al.* [19] to perform a frequency-based decoupling of the structure and by Voormeeren *et al.* [20], who studied the uncertainty propagation when performing a frequency-based substructuring. However, the classic method [15] is still being used, for example, for frequency decoupling [21] and for the study of common issues arising during the experimental approach to frequency-based substructuring [22, 23].

As presented, for example, by Salvini *et al.* [24], besides the analysis of the modifications, which change the structure's mass and stiffness properties, the SMURF can also be used to predict the influence of additional constraints added to the structure by knowing the frequency response of the unconstrained structure only. Furthermore, Avitabile [25] compared the mode shapes of the additionally constrained structure obtained using the SMURF method with the analytical mode shapes. With the introduction of the dual-domain decomposition method [18], the definition of the constraints can be given with a boolean constraint matrix.

In [18, 24, 25] the authors assumed a zero-response for the constraints and focused on the response of the constrained structure to the force excitation.

This research presents a SMURF technique, where the constrained degrees of freedom (DOFs) are excited with known kinematics. With this, it is possible to predict the frequency response of the constrained structure using the SMURF method, when the excitation is given in terms of the base motion as well as the excitation force. This research predicts the displacement and the stress response of the kinematically excited structure where only the natural frequencies and mode shapes of the unconstrained structure are required.

This manuscript is organized as follows. In Section 2 the theory of the extended SMURF technique is given, which can be used for the kinematically excited structures. In Section 3 this method is applied to a simple beam and to a larger finite-element model, where the influence of the modal truncation error, the matrix singularity and the effect of structural damping are discussed. The last section presents the conclusions.

2. Theoretical background

Most often the dynamic response of the structure is observed when it is excited with an applied force. When a structure is excited with a given base motion (*e.g.*, during vibration fatigue testing, earthquake) the information about the force applied to the constraints is not available. Consequently, the basic theory of the structure's response to the base motion differs from the classic force-excitation approach [8]. In this Section the main differences between the force- and base-excitation responses are shown that indicate the motivation for the study that follows. In it, the theory of the extended SMURF technique is deduced with respect to the applied base excitation.

2.1. Displacement response to the force excitation

When a viscously damped MDOF dynamic structure is excited with an applied force the equilibrium equations can be written as [8]:

$$\mathbf{M}\,\ddot{\mathbf{x}} + \mathbf{C}\,\dot{\mathbf{x}} + \mathbf{K}\,\mathbf{x} = \mathbf{f}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are mass, damping and stiffness matrices, respectively. $\mathbf{x}$ denotes the vector of total displacements and $\mathbf{f}$ denotes the excitation force vector. The solution to the Eq. (1) is given as the relation between excitation forces and response DOFs:

$$\mathbf{x} = \mathbf{H}(\omega)\,\mathbf{f}, \tag{2}$$

where ω denotes frequency and $\mathbf{H}(\omega)$ denotes the structure's receptance matrix. The jk-th element of the receptance matrix $\mathbf{H}(\omega)$ gives the structure's total response at the j-th degree of freedom when the structure is excited at the k-th degree of freedom. Accordingly, the jk-th element of the receptance matrix $\mathbf{H}(\omega)$ can be calculated as [8]:

$$H_{jk}(\omega) = \sum_{r=1}^{N} \left(\frac{\phi_{jr}\,\phi_{kr}}{\omega_r\,\xi_r + \mathrm{i}\left(\omega - \omega_r\sqrt{1-\xi_r^2}\right)} + \frac{\phi_{jr}^*\,\phi_{kr}^*}{\omega_r\,\xi_r + \mathrm{i}\left(\omega - \omega_r\sqrt{1-\xi_r^2}\right)} \right), \tag{3}$$

where ω_r denotes the r-th natural frequency, ξ_r the damping loss factor of the r-th natural frequency, ϕ_{jr} the jr-th element of the mass-normalised modal matrix $\boldsymbol{\phi}$ and * a complex conjugation. Often, damping loss factors are given in terms of the Rayleigh proportional damping factors α and β and for the case of the viscous damping model the following relation holds [8]:

$$\xi_r = \frac{\alpha}{2\,\omega_r} + \frac{\beta\,\omega_r}{2}. \tag{4}$$

2.2. Displacement response to the base excitation

An alternative possibility to excite the dynamic structure is to assign a certain kinematic motion at the given DOF. Fig. 1 shows a typical base-excitation setup, where the base displacement is denoted by $x_0(t)$. The vector of relative displacements $\mathbf{y}(t)$ is [10]:

$$\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{g}\, x_0(t), \tag{5}$$

where $\mathbf{x}(t)$ is the vector of the total displacements and $\mathbf{g}$ is the direction vector of the structure's degrees of freedom.

For a steady-state response to the base excitation the equations of motion of the viscously damped structure are [10]:

$$\left(\mathbf{K} - \omega^2\, \mathbf{M} + \mathrm{i}\, \omega\, \mathbf{C}\right)\, (\mathbf{x} - x_0\, \mathbf{g}) = \omega^2\, x_0\, \mathbf{M}\, \mathbf{g}. \tag{6}$$

Since $(\mathbf{x} - x_0\, \mathbf{g}) = 0$ holds for the DOFs at the excitation base it is obvious that in terms of relative displacements the homogenous form of Eq. (6) provides eigenvalues (*i.e.*, natural frequencies and mode shapes) identical to the eigenvalues of the structure with a fixed base. Therefore, we can define ${}^{\mathbf{c}}\mathbf{H}(\omega) = (\mathbf{K} - \omega^2\, \mathbf{M} + \mathrm{i}\, \omega\, \mathbf{C})^{\mathbf{-1}}$ as the receptance matrix [10] of the constrained structure with a fixed base, where ${}^{\mathbf{-1}}$ denotes the inverse matrix. Eq. (6) can be rewritten in the form of the transmissibility matrix $\mathbf{Q}(\omega)$ [10]:

$$\mathbf{Q}(\omega) = \frac{\mathbf{y}}{x_0} = \omega^2\, {}^{\mathbf{c}}\mathbf{H}(\omega)\, \mathbf{M}\, \mathbf{g}. \tag{7}$$

From this equation it is evident that the mass matrix $\mathbf{M}$ and the direction vector $\mathbf{g}$ must be known in order to predict the structure's transmissibility in the case of a base excitation. $\mathbf{M}$ and $\mathbf{g}$ are not always attainable, especially when only the response model or an incomplete modal model of the structure are available. To overcome the problem of an unknown scaling of ($\mathbf{M}\,\mathbf{g}$) for the transmissibility matrix, Ewins [10] suggests a correction using an experimentally obtained calibration factor. Additionally, in studies by Razeto *et al.* [1] and Mayes and Bridgers [26] different methods for mass-scaling using experimental base-excitation testing are presented. In any case, when only the modal properties of the constrained structure are known, an additional step is required to predict the structure's transmissibility for base excitation, whether by obtaining the spatial properties $\mathbf{M}$ and $\mathbf{g}$ or by additional experimental work.

This study presents a new method for obtaining the base-excitation response from the structure's modal properties only (*e. g.,* natural frequencies, mode shapes and damping loss factors), therefore making the spatial properties ($\mathbf{M}\,\mathbf{g}$) or additional experimental measurements redundant. The method introduced is based on the structural modification [25] and is presented in the following section.

2.3. Structural modification using response functions

One of the impedance-based methods for performing the structural modification is the SMURF (*structural modification using response functions*) method [15]. The SMURF method was already used to predict a response to the force excitation of a structure with additional constraints [13, 24, 25]. In this research a new evaluation of the SMURF method is introduced that includes the imposed kinematics of the added constraints.

2.3.1. Constrained and unconstrained structure

As indicated with a simple beam in Sec. 2.2 the modal model of the base-excited structure is equal to the modal model of the structure with a fixed base. A new approach introduced in this study enables a prediction of the response to base excitation from the response of the unconstrained structure to the force excitation only.

When performing the modification analysis with the SMURF technique either the response model or the (incomplete) modal model of the initial structure is considered to be known and the response of the modified structure is being searched for. With a structural modification we introduce additional constraints to the initial structure, to which the base excitation is applied. Similarly, in this deduction the unconstrained structure (Fig. 2(a), denoted with $^{\mathbf{u}}$) is regarded as an initial state and the constrained, base-excited structure (Fig. 2(b), denoted with $^{\mathbf{c}}$) is regarded as the modified structure.

Let us divide the unconstrained structure's (Fig. 2(a)) DOFs into two groups: the constrained c and the unconstrained u DOFs. The DOF is regarded as constrained if the base excitation is applied directly to it. Accordingly, the response of the structure can be rewritten as:

$$\left\{ \begin{array}{c} \mathbf{x_c} \\ \mathbf{x_u} \end{array} \right\} = \left[\begin{array}{cc} ^{\mathbf{u}}\mathbf{H_{cc}} & ^{\mathbf{u}}\mathbf{H_{cu}} \\ ^{\mathbf{u}}\mathbf{H_{uc}} & ^{\mathbf{u}}\mathbf{H_{uu}} \end{array} \right] \left\{ \begin{array}{c} \mathbf{f_c} \\ \mathbf{f_u} \end{array} \right\}, \tag{8}$$

where $\mathbf{x_c}$ and $\mathbf{x_u}$ denote the response of the constrained and unconstrained DOFs, respectively. Eq. (8) presents the basic idea of the SMURF technique.

Commonly, the authors [24, 25] at this point introduce an assumption $\mathbf{x_c} = 0$, therefore neglecting the base-excitation component and focusing on the force excitation response. In this research the response of the constrained DOFs is regarded as non-zero and known, which complies with the concept of vibration testing.

2.3.2. Displacement response to the base excitation using SMURF

When a base excitation is applied to the constrained DOFs the condition $\mathbf{x_c} = \mathbf{x_0}$ holds, and by refactoring Eq. (8) it follows:

$$\mathbf{x_0} = {}^{\mathbf{u}}\mathbf{H_{cc}}\,\mathbf{f_c} + {}^{\mathbf{u}}\mathbf{H_{cu}}\,\mathbf{f_u}, \tag{9}$$

$$\mathbf{x_u} = {}^{\mathbf{u}}\mathbf{H_{uc}}\,\mathbf{f_c} + {}^{\mathbf{u}}\mathbf{H_{uu}}\,\mathbf{f_u}. \tag{10}$$

From Eq. (9) the unknown force $\mathbf{f_c}$ can be written as:

$$\mathbf{f_c} = {}^{\mathbf{u}}\mathbf{H_{cc}^{-1}}\,\mathbf{x_0} - {}^{\mathbf{u}}\mathbf{H_{cc}^{-1}}\,{}^{\mathbf{u}}\mathbf{H_{cu}}\,\mathbf{f_u}. \tag{11}$$

From Eqs. (10) and (11) the absolute response of the constrained structure follows:

$$\mathbf{x_u} = \left({}^{\mathbf{u}}\mathbf{H_{uc}}\,{}^{\mathbf{u}}\mathbf{H_{cc}^{-1}}\right)\mathbf{x_0} + \left({}^{\mathbf{u}}\mathbf{H_{uu}} - {}^{\mathbf{u}}\mathbf{H_{uc}}\,{}^{\mathbf{u}}\mathbf{H_{cc}^{-1}}\,{}^{\mathbf{u}}\mathbf{H_{cu}}\right)\mathbf{f_u}. \tag{12}$$

Considering the relative displacement definition in Eq. (5) the relative response of the base-excited structure can be written in the final form as:

$$\mathbf{y} = \underbrace{\left({}^{\mathbf{u}}\mathbf{H_{uc}}\,{}^{\mathbf{u}}\mathbf{H_{cc}^{-1}} - \widetilde{\mathbf{g}}\right)}_{\mathbf{Q}(\omega)}\mathbf{x_0} + \underbrace{\left({}^{\mathbf{u}}\mathbf{H_{uu}} - {}^{\mathbf{u}}\mathbf{H_{uc}}\,{}^{\mathbf{u}}\mathbf{H_{cc}^{-1}}\,{}^{\mathbf{u}}\mathbf{H_{cu}}\right)}_{{}^{\mathbf{c}}\mathbf{H}(\omega)}\mathbf{f_u}. \tag{13}$$

In Eq. (13) $\widetilde{\mathbf{g}}$ is defined as:

$$\widetilde{\mathbf{g}} = \begin{bmatrix} g_1/N_C & \cdots & g_1/N_C \\ \vdots & \ddots & \vdots \\ g_{N_U}/N_C & \cdots & g_{N_U}/N_C \end{bmatrix}, \tag{14}$$

where g_i is the i-th element of the vector $\mathbf{g}$ and N_C and N_U are a number of constrained and unconstrained DOFs, respectively. The shape of the matrix $\widetilde{\mathbf{g}}$ is $[N_U,\, N_C]$.

Eq. (13) gives a full description of the structure with additional constraints for the force $\mathbf{f_u}$ and also for the base excitation $\mathbf{x_0}$. If Eq. (13) is compared with Eq. (7) it is clear that the extended SMURF technique introduces the calculation of the scaled transmissibility function $\mathbf{Q}\,(\omega)$ to the base excitation without any information about the mass matrix of the structure. When the Eq. (13) is used for the calculation of the relative displacement response to the base excitation only the modal model of the unconstrained structure is required to obtain matrices ${}^{\mathbf{u}}\mathbf{H_{uc}}$ and ${}^{\mathbf{u}}\mathbf{H_{cc}}$ with Eq. (3). This greatly reduces the time and amount of data needed to compute the response to the base excitation compared to the solving the coupled matrix equations (Eq. (6)) for discrete excitation frequencies used by finite-element software. Additionally, for the calculation of the displacement response to the base excitation only the direct point receptance matrix ${}^{\mathbf{u}}\mathbf{H_{cc}}$ and the receptance matrix ${}^{\mathbf{u}}\mathbf{H_{uc}}$ between the constrained and unconstrained DOFs must be known. When the displacement response of the constrained structure ${}^{\mathbf{c}}\mathbf{H}(\omega)$ to the excitation force $\mathbf{f_u}$ is investigated, the additional information of the receptance ${}^{\mathbf{u}}\mathbf{H_{uu}}$ between unconstrained DOFs should be available.

2.3.3. Stress response to the base excitation using SMURF

An internal mechanical stress is important for a fatigue-life estimation during vibration tests [27]. Therefore, a similar deduction to the one in Sec. 2.3.2 can be proposed to obtain the stress transmissibility of the constrained structure from the stress transmissibility of the unconstrained structure. The relation between the excitation force and the stress of the unconstrained structure can be defined as:

$$\boldsymbol{\sigma} = {}^{\mathbf{u}}_{\boldsymbol{\sigma}}\mathbf{H}(\omega)\,\mathbf{f}, \tag{15}$$

where ${}^{\mathbf{u}}_{\boldsymbol{\sigma}}\mathbf{H}(\omega)$ is the stress transmissibility matrix of the unconstrained structure. The stress transmissibility function can be obtained from Eq. (3), where the the elements ϕ_{jr} and ϕ_{kr} of the modal matrix $\boldsymbol{\phi}$ describe stress values at certain mode shapes instead of the displacement or rotation values. With a known stress transmissibility we can deal with vibrational fatigue in case of a random or harmonic excitation.

An equation similar to Eq. (8) can be written for the stress response; however, because the additional constraints do not imply known stress values for constrained DOFs it is not reasonable to separate the stress $\boldsymbol{\sigma}$ into $\boldsymbol{\sigma}_{\mathbf{c}}$ and $\boldsymbol{\sigma}_{\mathbf{u}}$. Consequently, the stress response Eq. (15) can be rewritten as:

$$\boldsymbol{\sigma} = \left[\begin{array}{cc} {}^{\mathbf{u}}_{\boldsymbol{\sigma}}\mathbf{H}_{\mathbf{c}} & {}^{\mathbf{u}}_{\boldsymbol{\sigma}}\mathbf{H}_{\mathbf{u}} \end{array}\right] \left\{\begin{array}{c} \mathbf{f}_{\mathbf{c}} \\ \mathbf{f}_{\mathbf{u}} \end{array}\right\}. \tag{16}$$

Considering the definition of the substitute force (Eq. (11)) into Eq. (16) the stress response of the constrained structure can be obtained with:

$$\boldsymbol{\sigma} = \left({}^{\mathbf{u}}_{\boldsymbol{\sigma}}\mathbf{H}_{\mathbf{c}}\,{}^{\mathbf{u}}\mathbf{H}_{\mathbf{cc}}{}^{-1}\right)\,\mathbf{x}_{\mathbf{0}} + \left({}^{\mathbf{u}}_{\boldsymbol{\sigma}}\mathbf{H}_{\mathbf{u}} - {}^{\mathbf{u}}_{\boldsymbol{\sigma}}\mathbf{H}_{\mathbf{c}}\,{}^{\mathbf{u}}\mathbf{H}_{\mathbf{cc}}{}^{-1}\,{}^{\mathbf{u}}\mathbf{H}_{\mathbf{cu}}\right)\,\mathbf{f}_{\mathbf{u}}. \tag{17}$$

Interestingly, to predict the stress response of the constrained structure to the base or force excitation we must have information about ${}_{\boldsymbol{\sigma}}\mathbf{H}$ and $\mathbf{H}$, *i.e.*, the stress and displacement transmissibility functions, respectively. However, when only the base excitation $\mathbf{x}_{\mathbf{0}}$ is applied to the structure ($\mathbf{f}_{\mathbf{u}} = 0$), the stress response at a particular location is obtained by knowing the stress transmissibility ${}^{\mathbf{u}}_{\boldsymbol{\sigma}}\mathbf{H_c}$ between the constrained DOFs and the unconstrained locations and the direct point transmissibility matrix ${}^{\mathbf{u}}\mathbf{H_{cc}}$ of the constrained DOFs, only. The two methods presented in Eqs. (13) and (17) can be used in a numerical or experimental analysis of the dynamic system.

An additional advantage of the presented method is the significant reduction in the calculation time when compared to the time needed to perform a harmonic analysis of the structure subjected to a base excitation. This is even more so when the stress response is obtained with the finite-element method, where the stress response for each discrete frequency point is calculated by expanding the displacement response solution that additionally extends the calculation time.

2.3.4. Singularity of direct point receptance matrix

The presented SMURF method has an important numerical limitation when the matrix ${}^{\mathbf{u}}\mathbf{H_{cc}}$ (Eqs. (13) and (17)) is ill-conditioned [11, 17]. This occurs when c DOFs of the unconstrained structure (Fig. 2) are located on a region with high local stiffness, thus have similar response functions which appear in ${}^{\mathbf{u}}\mathbf{H_{cc}}$ matrix as linearly dependent rows. Inverting the ill-conditioned ${}^{\mathbf{u}}\mathbf{H_{cc}}$ results in erroneous response prediction of the modified structure. Urgueira [11] addressed this problem by identifying the independent DOFs using singular value decomposition and reducing the shape of the ${}^{\mathbf{u}}\mathbf{H_{cc}}$ matrix to the $[r \times r]$, where r is the rank of ${}^{\mathbf{u}}\mathbf{H_{cc}}$ and, coincidentally, a number of independent coordinates. However, the rank of ${}^{\mathbf{u}}\mathbf{H_{cc}}$ depends on the chosen threshold value and cannot be uniquely determined for practical cases.

In this study the fixed surfaces of the constrained structure are assumed to be rigid and the c DOFs are described with the base motion $\mathbf{x_0}$. Logically, there are no interactions between the fixed c DOFs of the constrained structure. Assuming this, there is no need to obtain the interactions between c DOFs of the unconstrained structure. Additionally, the number of mutually independent non-zero elements of the excitation vector $\mathbf{x_0}$ equals to the number of excitation directions n_{exc} in which the base excitation is applied. If the rank r of the matrix $^{\mathbf{u}}\mathbf{H_{cc}}$ equals to the number of excitation directions n_{exc} then the matrix $^{\mathbf{u}}\mathbf{H_{cc}}$ and vector $\mathbf{x_0}$ can be reduced to the sizes of $[n_{\mathrm{exc}} \times n_{\mathrm{exc}}]$ and $[n_{\mathrm{exc}} \times 1]$, respectively. With an appropriate shape reduction the matrix $^{\mathbf{u}}\mathbf{H_{cc}}$ becomes well-conditioned and therefore eliminates the problem of matrix inversion. However, to ensure that $r = n_{\mathrm{exc}}$ the rigid connection between c DOFs of the unconstrained structure must be applied along with the additional constraints of c DOFs in non-excited directions. The presented approach of the $^{\mathbf{u}}\mathbf{H_{cc}}$ rank reduction to the fixed value of $r = n_{\mathrm{exc}}$ is demonstrated in a numerical experiment in Section 3.2.

As stated, an assumption of rigid fixation must be fulfilled. For the case of numerical evaluation that is performed in this research the rigid fixation can be easily applied. However, experimentally it is hardly possible to assure an ideal rigid fixation therefore one should be aware of the possible problems that arise due to the non-rigid fixation [12].

3. Numerical experiment

The numerical analyses of the extended SMURF technique, for a calculation of the displacement and stress response to the base excitation, and introduced in Sec. 2, is presented here for two different numerical examples.

Firstly, a simple beam with a rectangular cross-section is researched to compare the results obtained with the SMURF approach introduced here to those obtained by solving the equilibrium matrix equation (6) for each frequency of the excitation. Secondly, a Y-shaped structure with two weights and a hole is used to show the applicability of the method for larger numerical models, where the singularity of the direct point transmissibility ${}^{\mathbf{u}}\mathbf{H}_{\mathbf{cc}}$ is present. In both cases the analysis will be focused on the transmissibility function $\mathbf{Q}(\omega)$ to base excitation, which is the original contribution of this research.

3.1. Simple beam

Two cases of a beam with a rectangular cross-section of 10 mm $\times$ 30 mm and with a length of 800 mm are analyzed. In the first case (A, Fig. 3) the additional constraints $y(l = 0\,\mathrm{mm}) = 0$ and $\mathrm{d}y(l = 0\,\mathrm{mm})/\,\mathrm{d}l = 0$ are introduced at one position only. Based on this simple modification of the beam, the consistency of the extended SMURF for the base-excited structure is presented.

In the second case (B, Fig. 4) the beam is constrained at three distinct locations (at the left end with $y(l = 0\,\mathrm{mm}) = 0$ and $\mathrm{d}y(l = 0\,\mathrm{mm})/\,\mathrm{d}l = 0$, 40 mm from the left end with $y(l = 40\,\mathrm{mm}) = 0$ and at the right end with with $y(l = 800\,\mathrm{mm}) = 0$); this modification is proposed to show the general applicability of the extended SMURF method to an arbitrary set of additional constraints. In case B the SMURF method was applied to the free-free beam (B1) and to the beam with initial constraint (B2) described with $\mathrm{d}y(l{=}0\,\mathrm{mm})/\,\mathrm{d}l{=}0$. In this way the influence of the number of added constraints to the transmissibility function of the modified structure can be researched.

The numerical model of the beam was made from 20 beam elements of length 40 mm with three DOFs at each node. The beam was considered undamped (the influence of damping is studied later). In the numerical example of the beam all the natural frequencies and mode shapes were used for the SMURF technique to avoid the effect of truncation error [25].

In both cases of fixation (Figs. 3 and 4) the transmissibility in terms of displacement was observed at the node 20 ($l = 760\,\mathrm{mm}$) and the transmissibility in terms of stress at the node 10 ($l = 360\,\mathrm{mm}$). From Figs. 5 and 6 it is clear that in the modification case A the absolute and relative displacement transmissibility of the modified structure coincide with the reference value and, therefore, show the appropriateness of Eqs. (12) and (13). Additionally, the predicted stress-transmissibility function, shown in Fig. 7, is in good accordance with the theoretical stress transmissibility, obtained by a harmonic analysis of a finite-element model with the given base excitation. The relative error, shown in Figs. 5 - 7 can be assigned to small discrepancies of the natural frequencies and anti-resonances that inherently lead to large errors in resonant and anti-resonant areas.

A good prediction of the resonance and anti-resonance values was expected, since all the modes were included in the initial transmissibility calculation with no truncation error. The presented numerical example shows an accurate prediction of the transmissibility values of the base-excited structure, obtained without any information about the mass matrix or the experimentally derived scaling factor.

The predicted transmissibility of the beam modification case B) is shown in Figs. 8 and 9. By comparing the predicted and theoretical transmissibility it is evident that the SMURF method gives a good estimation, even when the base excitation is applied at several locations. This conclusion is applicable when the transmissibility functions of the constrained DOFs are not alike and consequently the response matrix $^{\mathbf{u}}\mathbf{H}_{\mathbf{cc}}$ is not singular. By observing the relative error in Fig. 8 or the absolute displacement transmissibility near the first natural frequency (Fig. 9) it is evident that the method gives more accurate results when the number of additional constraints is lower, regardless of the truncation error.

3.2. Y-shaped structure

To demonstrate some practical problems with of the use of the SMURF method for the prediction of the structure's stress transmissibility to the base excitation a Y-shaped specimen, shown in Fig. 10, is analyzed. The specimen was custom designed to perform the vibrational fatigue tests [28]. Since the vibrational fatigue life is directly related to the stress profile at the critical location, only the stress transmissibility will be studied. Here, the Y-shaped specimen is regarded as a damped, linear structure and the numerical model is used to obtain the natural frequencies and mode shapes.

The $^{\mathbf{u}}\mathbf{H}_{\mathbf{cc}}$ reduction approach, introduced in Sec. 2.3.4, was adopted for the transmissibility prediction of the base-excited Y-shaped specimen, fixed at the surface marked in Fig. 10 and excited in all three translational DOFs; therefore $\mathbf{x_0} = [1, 1, 1]^T$. The von Mises equivalent stress at the location, marked in Fig. 10, was considered as an observed stress-response quantity. The analysis of the Y-shaped specimen demonstrates the applicability of the new method to larger numerical models. Furthermore, two additional aspects are studied: the influence of $^{\mathbf{u}}\mathbf{H}_{\mathbf{cc}}$ matrix reduction and the influence of the structural damping on the stress-transmissibility prediction.

First, the influence of $^{\mathbf{u}}\mathbf{H}_{\mathbf{cc}}$ matrix reduction on avoiding the singularity was observed. In order to avoid a deviation of the prediction as a result of the damping influence, the structure was, in this case, regarded as undamped and the first 30 natural frequencies and mode shapes were used for the SMURF method. The transmissibility thus obtained is compared with the theoretical transmissibility in Fig. 11, from which we can conclude that the reduction of the $^{\mathbf{u}}\mathbf{H}_{\mathbf{cc}}$ does not introduce a significant error to the SFRF prediction.

However, structural damping is always present and must be considered when the stress transmissibility of the structure is used for a calculation of the vibration fatigue life. Three different sets of proportional damping factors were used: high damping ($[\ \alpha,\ \beta\ ] = [\ 0,\ 5{\cdot}10^{-7}\ ]$), low damping ($[\ \alpha, \beta\ ] = [\ 0,\ 1{\cdot}10^{-7}\ ]$) and no damping ($[\ \alpha, \beta\ ] = [\ 0, 0\ ]$). In Fig. 12 the transmissibility functions for all three damping sets are shown.

From observing the transmissibility near the first natural frequency it is evident that the amplitudes of the predicted transmissibility agree very well with the theoretical values. Nevertheless, the truncation error is shown as the frequency shift between the theoretical and the predicted natural frequency and is present in the case of the damped structure as well. The prediction of the transmissibility of the damped Y-specimen near the fifth natural frequency is less accurate. This can be attributed to the higher truncation error. From the observation of transmissibility near the fifth natural frequency it can be concluded that the inclusion of damping into the SMURF method does not introduce a significant additional error to the transmissibility prediction.

This observation highlights another advantage of the developed technique. The definition of the structure's receptance Eq. (3) introduces a possibility to apply the damping in the form of proportional or modal damping. When the transmissibility in the case of base excitation is obtained by solving the equilibrium equation (6), only proportional damping can be applied [8]. With the presented method of the base-excitation transmissibility calculation the damping can be given in terms of modal damping for each natural frequency without any additional error being introduced.

When comparing the calculation times required to compute the stress transmissibility to the base excitation the new SMURF method proves to be significantly faster. In the case of the Y-shaped specimen, where the frequency range was given as 0 Hz to 3000 Hz, with a 1-Hz frequency resolution, the extended SMURF method takes 7 seconds, compared to the method implemented in commercial FEM software, which takes 240 seconds.

4. Conclusions

In this research a structural modification using frequency-response functions has been applied and extended to predict the transmissibility of the constrained structure when it is kinematically excited by base motion. With the presented method any spatial DOF or stress value can be regarded as the response quantity. For the prediction of the transmissibility in the case of base excitation only the natural frequencies and mode shapes are required; this makes this method convenient when no information about the mass and the stiffness matrices is available.

The developed method is applied to a numerical experiment and the transmissibility prediction is very good for the displacement as well as for the stress response. Furthermore, due to the simple calculation of the transmissibility functions of the unconstrained structure the estimation of the structure's response can be fast, when compared to the harmonic analysis of the finite-element model. This can prove to be convenient when a large number of frequency points are required for the response description.

Finally, the method is also applicable in cases when structural damping is present. The damping can be given in terms of proportional or modal damping, therefore increasing the accuracy of the transmissibility near natural frequencies. This is a valuable advantage when the stress response is used to estimate the vibrational fatigue life, since the response amplitude near natural frequencies introduces the most damage to the structure.

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Figures

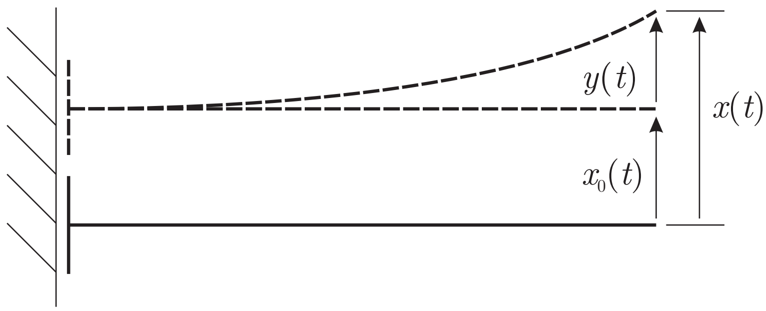

Figure 1: Base excitation of the cantilever beam with denoted displacements at the cantilever tip.

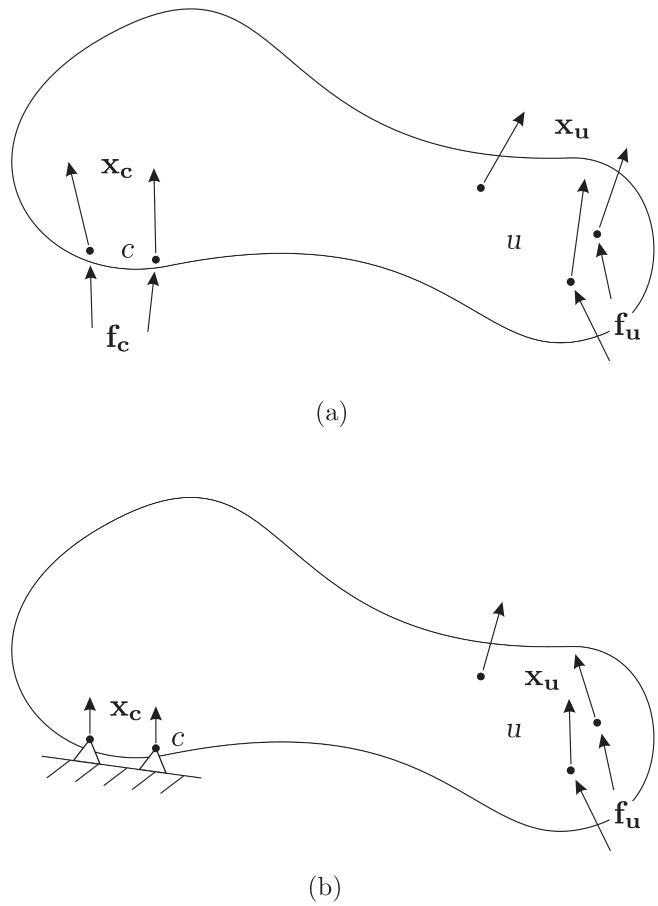

Figure 2: a) Unconstrained and b) constrained structure.

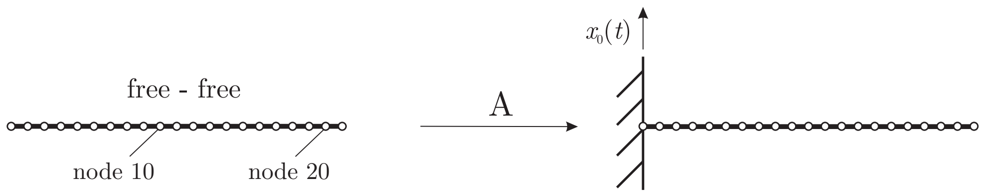

Figure 3: Case A) beam with an added constraint at one location.

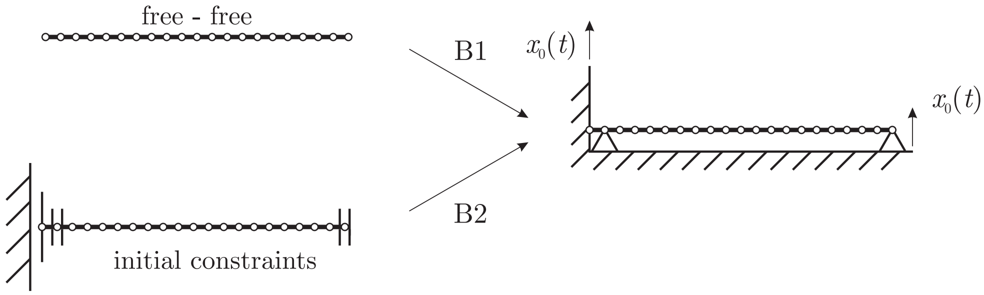

Figure 4: Case B) beam with added constraints at three locations.

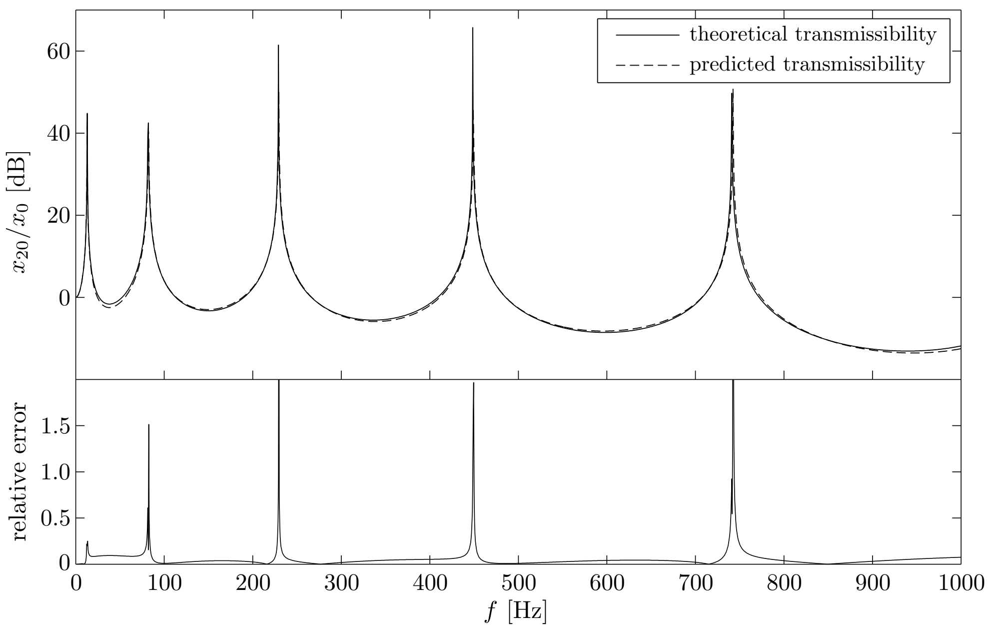

Figure 5: Case A; Absolute displacement transmissibility x_{20}/x_0 and a relative error.

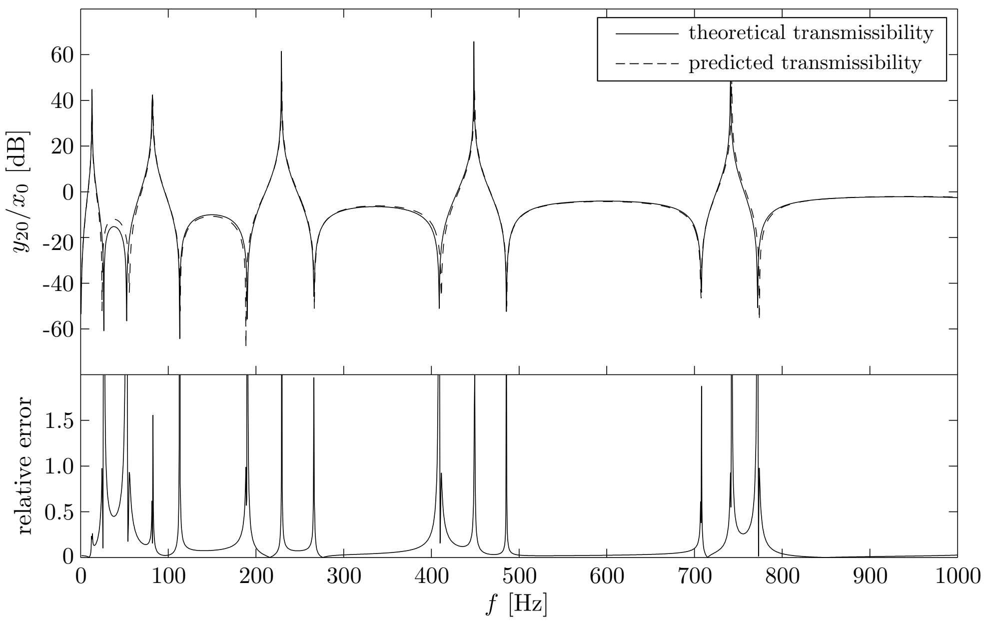

Figure 6: Case A; Relative displacement transmissibility y_{20}/x_0 and a relative error.

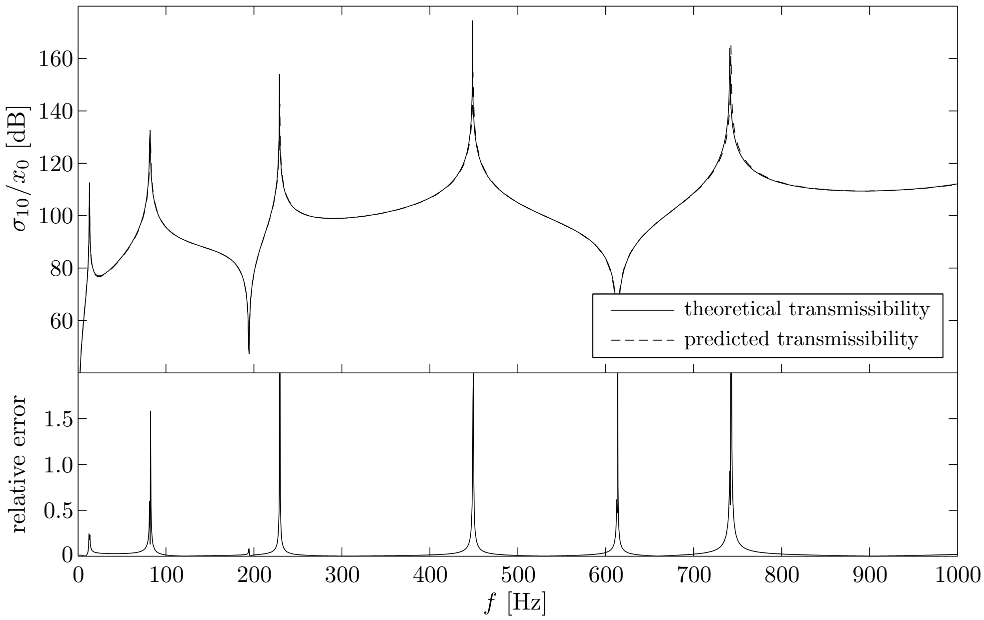

Figure 7: Case A; Stress transmissibility σ_{10}/x_0 and a relative error.

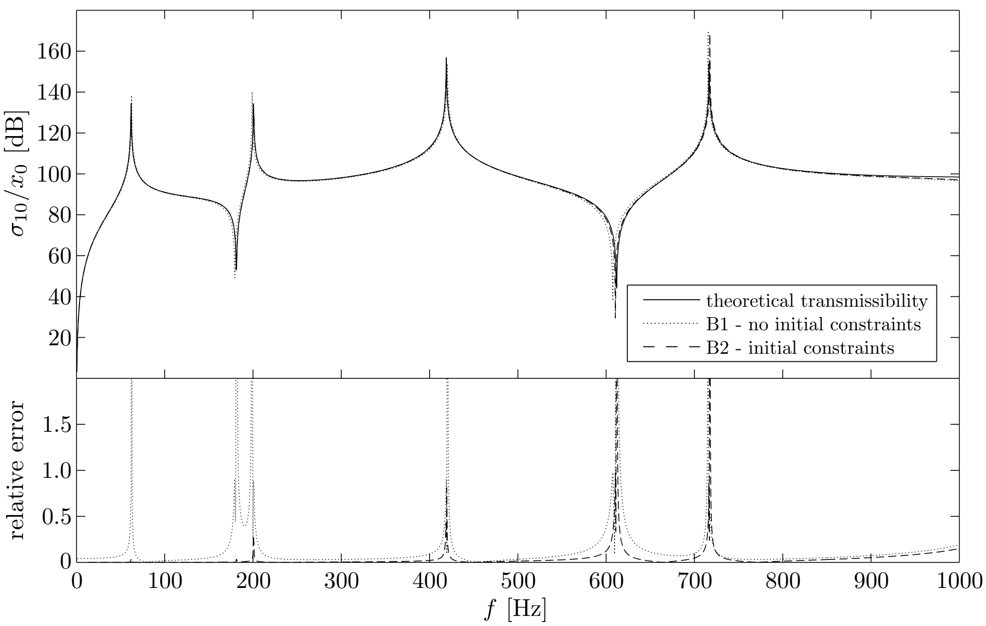

Figure 8: Case B; Stress transmissibility σ_{10}/x_0 and a relative error.

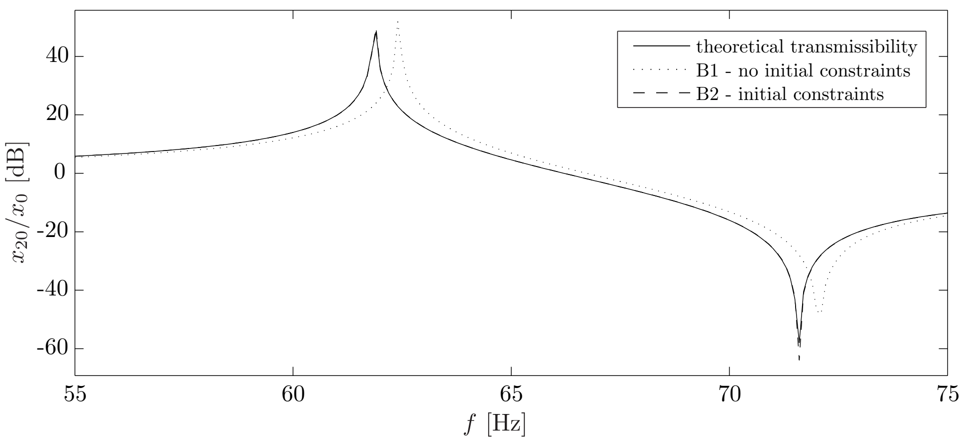

Figure 9: Case B; the error of modification method at first natural frequency.

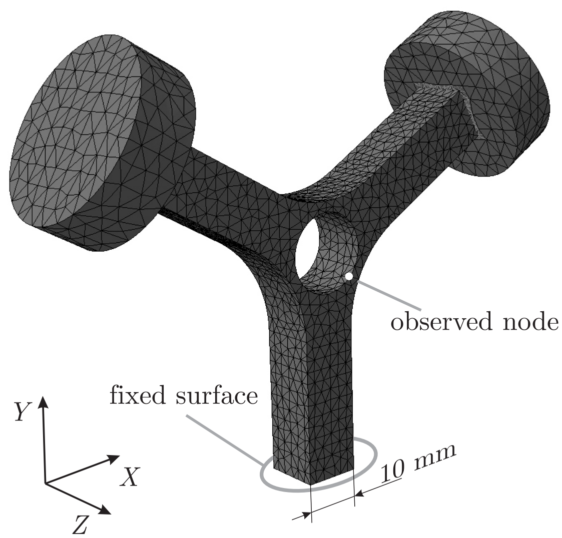

Figure 10: Model of Y-shaped specimen.

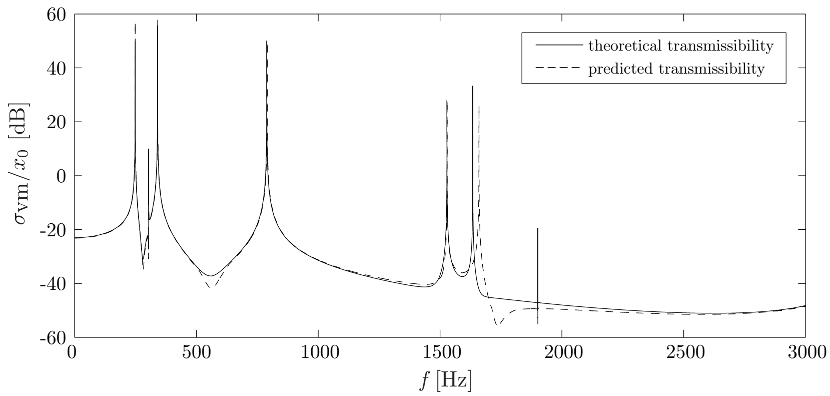

Figure 11: Stress transmissibility of the Y-shaped specimen with neglected damping.

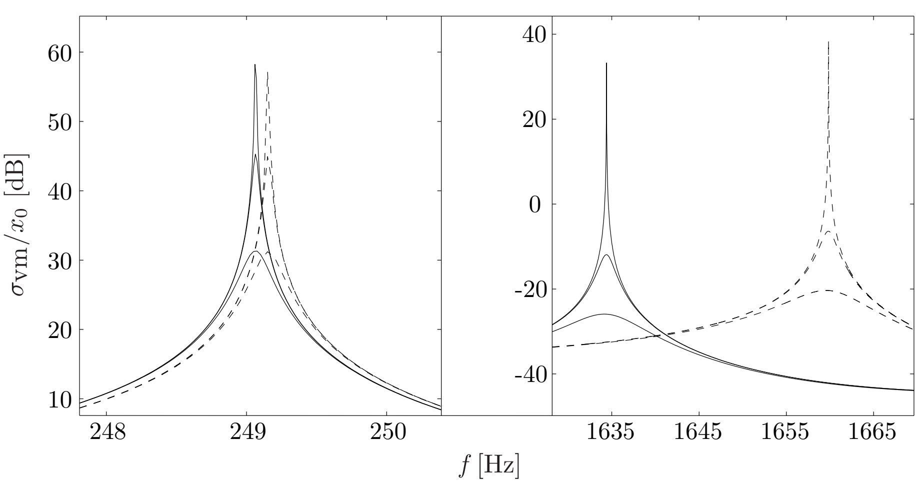

Figure 12: Stress transmissibility of a Y-shaped specimen with high, low and no damping close to the first and the fifth natural frequency; (—) theoretical and (- - -) predicted transmissibility.

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