

Vibration fatigue using modal decomposition

Matjaž Mršnik, Janko Slavič*, Miha Boltežar

Faculty of Mechanical Engineering, University of Ljubljana, Slovenia

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Matjaž Mršnik, Janko Slavič and Miha Boltežar

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Abstract

Vibration-fatigue analysis deals with the material fatigue of flexible structures operating close to natural frequencies. Based on the uniaxial stress response, calculated in the frequency domain, the high-cycle fatigue model using the S-N curve material data and the Palmgren-Miner hypothesis of damage accumulation is applied. The multiaxial criterion is used to obtain the equivalent uniaxial stress response followed by the spectral moment approach to the cycle-amplitude probability density estimation. The vibration-fatigue analysis relates the fatigue analysis in the frequency domain to the structural dynamics. However, once the stress response within a node is obtained, the physical model of the structure dictating that response is discarded and does not propagate through the fatigue-analysis procedure. The

*Corresponding author

Email address: `janko.slavic@fs.uni-lj.si` (Janko Slavič)

structural model can be used to evaluate how specific dynamic properties (*e.g.*, damping, modal shapes) affect the damage intensity. A new approach based on modal decomposition is presented in this research that directly links the fatigue-damage intensity with the dynamic properties of the system. It thus offers a valuable insight into how different modes of vibration contribute to the total damage to the material. A numerical study was performed showing good agreement between results obtained using the newly presented approach with those obtained using the classical method, especially with regards to the distribution of damage intensity and critical point location. The presented approach also offers orders of magnitude faster calculation in comparison with the conventional procedure. Furthermore, it can be applied in a straightforward way to strain experimental modal analysis results, taking advantage of experimentally measured strains.

Keywords: vibration fatigue, modal model, spectral moment method, multiaxial criterion

Nomenclature

${}_rA_{ij}^s$	stress modal constant
${}_r\mathbf{A}^s$	stress modal constants matrix
b	the Tovo-Benasciutti method parameter
C	S-N curve parameter
$\mathbf{C}$	stiffness tensor
d	damage intensity
d_{TB}	damage intensity, (Tovo-Benasciutti method)
$d_{\setminus M_r}$	damage intensity for the model excluding mode r
d_{NB}	damage intensity (narrowband approximation)
d_{SO}	damage intensity (Sakai-Okamura method)
D_r	modal damage intensity measure
$\mathbf{D}$	hysteretic damping matrix
$\overline{\mathbf{D}}$	displacements/strain field operator
$\mathbf{f}(t)$	time-dependent force excitation vector
$\mathbf{F}(\omega)$	frequency-dependent force excitation vector
$\mathbf{H}(\omega)$	receptance matrix
$\mathbf{H}_{sf}(\omega)$	stress frequency-response function
$\mathbf{H}_{sf}^r(\omega)$	modal stress frequency-response function
$\tilde{\mathbf{H}}_{sf}(\omega)$	stress frequency-response function for the reduced model
I_i^r	i -th order, r -th mode int. of ω -dep. part of stress resp.
k	S-N curve slope
$\mathbf{K}$	stiffness matrix
m	number of modes of the reduced modal model
m_i	i -th order spectral moment for the complete model
$\tilde{m}_i$	i -th order spectral moment for the reduced model

m_i^r	r -th mode i -th order spectral moment
$\mathbf{M}$	mass matrix
N	number of modes of the complete modal model
$\mathbf{Q}$	von Mises criterion constant coefficients matrix
$\tilde{S}_c(\omega)$	equivalent von Mises multiaxial stress
$S_c^r(\omega)$	equivalent modal von Mises multiaxial stress
$\mathbf{S}_{ff}(\omega)$	forced-excitation power spectral density
$\mathbf{S}_{ff,z}$	forced-excitation power spectral density acting along z -axis
$\mathbf{S}_{ss}(\omega)$	stress-response power spectral density
$\tilde{\mathbf{S}}_{ss}(\omega)$	stress-response power spectral density of the reduced model
$\mathbf{S}_{ss}^r(\omega)$	modal stress-response power spectral density
t	time
$\mathbf{x}(t)$	time-dependent vector of displacements
$\ddot{\mathbf{x}}(t)$	time-dependent vector of accelerations
$\mathbf{X}$	displacements amplitudes vector
$\mathbf{X}(\omega)$	frequency-dependent displacements amplitude vector
$\mathbf{X}_s(\omega)$	frequency-dependent stress responses
$\tilde{\mathbf{X}}_s(\omega)$	frequency-dependent stress response of reduced model

α_2	spectral width parameter
$\Gamma(\cdot)$	Euler gamma function
$\zeta_r(\omega)$	ω -dependent part of frequency-response function
η_r	damping loss factor
λ_r	r -th eigenvalue
ϕ_r	r -th mass normalized eigenvector
ϕ_r^s	r -th mass normalized stress modal shape

Φ	modal matrix (of mass-normalized eigenvectors)
ϕ_r^ε	r -th mass-normalized strain modal shape
Φ^s	stress modal matrix
ϕ_{ij}^s	element of stress modal matrix
ϕ_{ij}	element of modal matrix
ψ_r	r -th eigenvector
ω	angular frequency
ω_r	natural frequency

1. Introduction

Structures that vibrate intensely are prone to failure because of material fatigue. Different fatigue-analysis approaches are available to deal with such cases; vibration-fatigue analysis is one that is very suitable when the stress response occurs mainly as a consequence of rich structural dynamics [1, 2, 30].

In the course of the numerical analysis procedure, the vibration excitation is applied via the frequency-response functions (FRFs) to obtain the distribution of the stress (or strain) tensor in the frequency domain [3, 4] for the analyzed structure. Once the stress/strain tensor function of frequency at a particular material point is known, it has to be reduced to an equivalent uniaxial stress function, using a multiaxial criterion [1, 5–9], before subsequent vibration-fatigue methods can be applied. Such criteria are actively researched and recent comparative studies [10–14] have shown promising results. One of the simplest and most widely used is the frequency-domain equivalent von Mises (EVMS) criterion proposed by Preumont and Piéfort [6]. While EVMS provides relatively good results, there are some inconsistencies in its frequency-domain formulation, with regards

to the bending/torsion S-N curve data, as noted by Benasciutti [15]. Furthermore, Bonte et al. point out that the criterion disregards the phase angles between the tensor components [16].

Once the uniaxial equivalent is obtained using the multiaxial criterion, the cycle-amplitude distribution estimate is calculated. Finally, by means of the Palmgren-Miner rule [17, 18], the time-to-failure estimate [19, 20] is obtained. Based on the normal-distributed random-process definition [21] the spectral moments can be calculated in the frequency domain to estimate the cycle-amplitude distribution [22–24]. Several spectral methods have been developed, some of them very recently [23, 25, 26]. Comparison studies are available [23, 24, 26–29], highlighting the well-accepted Dirlik [22] approach as one of the most precise, closely matched in accuracy by the Benasciutti and Tovo spectral moment method [23].

Vibration fatigue analysis is thus an established procedure, performed in the frequency domain, starting with the power spectral density (PSD) profile of a random excitation, continuing with the FRF to the stress/strain multiaxial response, ending with the multiaxial fatigue criteria and spectral moment methods.

Dealing with stationary random loads, vibration fatigue analysis requires frequency-limited steady-state response solutions [6, 13, 14, 30, 31]. Therefore it can take full advantage of the computationally very efficient modal superposition approach, based on linear modal analysis, as opposed to the computationally more demanding direct time-integration methods [3]. Modal reduction is a common tool in structural analyses, used to reduce the numerical model to only a subset of modes [3, 32, 33]. There are a couple of recent studies [34–37] researching the modal approach to vibration fatigue, but very few [37] that take the full advantage of the available reduction

technique.

This research proposes an approach using the modal model and reduction technique to estimate the fatigue-damage intensity for a given random excitation profile. The fatigue damage is estimated directly through the properties of the modal model, offering a very direct and important relationship between the dynamic characteristics and the total damage. Applying modal reduction in the process further simplifies the procedure, opening up new opportunities for vibration-fatigue optimization. Additionally, fatigue analysis using the strain experimental modal analysis (EMA) results [4] can be performed in a very straightforward way, directly on the experimentally measured strains.

This research is organized as follows. The theoretical background for the standard structural dynamic analysis in modal space is given in Section 2. The new approach to vibration-fatigue analysis is proposed in Section 3. Section 4 presents a numerical experiment with the application of the proposed approach. Details of the method and its application are discussed in Section 5 and the conclusions are drawn in the final Section.

2. Structural dynamics

A hysteretically damped system with N degrees of freedom (DOF) is described by a system of differential equations [32]:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{iD}\dot{\mathbf{x}}(t) = \mathbf{f}(t), \quad (1)$$

where $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K}$ are the mass, hysteretic damping and stiffness matrices, $\mathbf{x}(t)$ is the displacement and $\ddot{\mathbf{x}}(t)$ is the acceleration vector. The excitation forces are denoted with $\mathbf{f}(t)$. In general, Eq. (1) presents a coupled system of differential equations. Assuming harmonic excitation $\mathbf{f}(t) = \mathbf{F}e^{i\omega t}$ with

an angular frequency ω and force amplitudes $\mathbf{F}$, and then further assuming a harmonic response $\mathbf{x}(t) = \mathbf{X} e^{i\omega t}$ with the response amplitudes $\mathbf{X}$, the expression from Eq. (1) is rewritten as:

$$(\mathbf{K} + i\mathbf{D} - \omega^2 \mathbf{M}) \mathbf{X} e^{i\omega t} = \mathbf{F} e^{i\omega t}, \quad (2)$$

which simplifies to:

$$(\mathbf{K} + i\mathbf{D} - \omega^2 \mathbf{M}) \mathbf{X} = \mathbf{F}. \quad (3)$$

In general, the excitation amplitudes $\mathbf{F}$ and the response amplitudes $\mathbf{X}$ are complex numbers, characterizing the amplitude as well as the phase delay. To emphasize the dependency of the amplitudes on the angular frequency ω , Eq. (3) is rewritten as:

$$\mathbf{X}(\omega) = \underbrace{(\mathbf{K} + i\mathbf{D} - \omega^2 \mathbf{M})^{-1}}_{\mathbf{H}(\omega)} \mathbf{F}(\omega) \quad (4)$$

where $\mathbf{H}(\omega)$ defines the receptance matrix [32]. The homogeneous part of Eq. (3) presents an eigensystem which, for the case of proportional damping, has complex eigenvalues λ_r :

$$\lambda_r^2 = \omega_r^2 (1 + i\eta_r), \quad (5)$$

where r is the eigenvalue/mode index and ω_r and η_r are the natural frequency and the damping loss factor, respectively. A particular eigenvalue is complementary to the eigenvector ψ_r . The mass-normalized eigenvector of mode r is denoted as ϕ_r ; they are stacked to form the mass-normalized modal matrix $\Phi = [\phi_1, \phi_2, \dots]$. Using this matrix, the receptance from Eq. (4) can be rewritten in a diagonal form (see [32] for details):

$$\mathbf{H}(\omega) = \Phi \left[\omega_r^2 (1 + i\eta_r) - \omega^2 \right]^{-1} \Phi^T \quad (6)$$

While the solution is thus obtained in the form of the displacements $\mathbf{X}(\omega)$, the steady-state stress response $\mathbf{X}_s(\omega)$ is required for the fatigue analysis. The mass-normalized stress modal shape (SMS) ϕ_r^s of the mode r can—under the assumption of small displacements—be obtained using Hooke’s law, *i.e.*, multiplying the stiffness tensor $\mathbf{C}$ with the r -th strain modal shape ϕ_r^ε [38]:

$$\phi_r^s = \mathbf{C} \phi_r^\varepsilon. \quad (7)$$

Where the r -th mass-normalized strain modal shape is obtained as:

$$\phi_r^\varepsilon = \overline{\mathbf{D}} \phi_r \quad (8)$$

and the operator $\overline{\mathbf{D}}$ is defined as [38]:

$$\overline{\mathbf{D}} = \frac{1}{2}(\nabla + \nabla^T) \quad (9)$$

where ∇ is the linear differential operator.

Applying the operator $\overline{\mathbf{D}}$ to the equation of motion (4) and then multiplying by $\mathbf{C}$ the stress amplitudes are [38–40]:

$$\mathbf{X}^s(\omega) = \mathbf{C} \overline{\mathbf{D}} \mathbf{X}(\omega) = \underbrace{\boldsymbol{\Phi}^s [\omega_r^2(1 + \mathbf{i} \eta_r) - \omega^2]^{-1} \boldsymbol{\Phi}^T}_{\mathbf{H}_{sf}(\omega)} \mathbf{F}(\omega). \quad (10)$$

$\mathbf{H}_{sf}(\omega)$ denotes the stress frequency-response function (FRF) for the complete modal model comprising all N modes:

$$\mathbf{H}_{sf}(\omega) = \sum_{r=1}^N \mathbf{H}_{sf}^r(\omega) = \sum_{r=1}^N \frac{{}_r\mathbf{A}^s}{\omega_r^2 - \omega^2 + \mathbf{i} \eta_r \omega_r^2}, \quad (11)$$

where $\mathbf{H}_{sf}^r(\omega)$ is the r -th modal stress FRF and an element of the stress modal constants matrix ${}_r\mathbf{A}^s$ is defined as [4], as similarly done by Preumont and Piéfort [6]:

$${}_rA_{ij}^s = \phi_{ir}^s \phi_{jr}, \quad (12)$$

ϕ_{ir}^s and ϕ_{jr} being the mass-normalized components of the modal matrices Φ^s and Φ , respectively.

Finally, the power spectral density (PSD) form is required [41] to describe the random excitation and response in the frequency domain:

$$\mathbf{S}_{ss}(\omega) = \mathbf{H}_{sf}(\omega) \cdot \mathbf{S}_{ff}(\omega) \cdot \mathbf{H}_{sf}^{*T}(\omega), \quad (13)$$

where $\mathbf{S}_{ss}(\omega)$ is the stress-response PSD and $\mathbf{S}_{ff}(\omega)$ is the force-excitation PSD.

3. Vibration-fatigue analysis for the reduced model

A new approach to vibration-fatigue analysis based on the reduced modal model is suggested in the following. The formulation is divided into three parts: stress-response calculation, multiaxial criterium application and damage-intensity estimation.

3.1. Reduced modal stress response

In modal superposition, the number of modes is always equal to the number of DOF for the complete model, which can be a very large number for an average FEM model. Usually, a frequency-domain solution is needed within a limited frequency range. Only a small subset m out of N modes can happen to participate within a limited band of interest; *e.g.* 0 to 2000 Hz is a common frequency range for automotive accelerated-vibration test procedures. In such cases a modal-reduction procedure can be performed, so that higher-frequency modes are truncated, as their contribution to the response is negligible. The reduced FRF, based on Eq. (11), is denoted by a tilde:

$$\tilde{\mathbf{H}}_{sf}(\omega) = \sum_{r=1}^{m < N} \mathbf{H}_{sf}^r(\omega), \quad (14)$$

and the PSD of the reduced stress response $\tilde{\mathbf{S}}_s(\omega)$ is then defined as:

$$\tilde{\mathbf{S}}_s(\omega) = \tilde{\mathbf{H}}_{sf}(\omega) \cdot \mathbf{S}_{ff}(\omega) \cdot \tilde{\mathbf{H}}_{sf}^{*\text{T}}(\omega). \quad (15)$$

If the modes participating in the response are well separated, then it might be reasonable to approximate the stress FRF PSD from Eq. (15) in the following way:

$$\tilde{\mathbf{H}}_{sf}(\omega) \cdot \mathbf{S}_{ff}(\omega) \cdot \tilde{\mathbf{H}}_{sf}^{*\text{T}}(\omega) \approx \sum_{r=1}^{m < N} \mathbf{H}_{sf}^r(\omega) \cdot \mathbf{S}_{ff}(\omega) \cdot \mathbf{H}_{sf}^{r* \text{T}}(\omega), \quad (16)$$

suggesting the single mode contributes significantly to the damage only in the close vicinity of its eigenfrequency. Substituting back into Eq. (13) and applying modal reduction, the expression for the reduced modal stress PSD is as follows:

$$\tilde{\mathbf{S}}_{ss}(\omega) = \sum_{r=1}^{m < N} \frac{{}_r\mathbf{A}_s \cdot \mathbf{S}_{ff}(\omega) \cdot {}_r\mathbf{A}_s^{*\text{T}}}{\omega^4 - 2\omega^2\omega_r^2 + (1 + \eta^2)\omega_r^4}, \quad (17)$$

Finally, Eq. (17) is rewritten as:

$$\tilde{\mathbf{S}}_{ss}(\omega) = \sum_{r=1}^{m < N} \zeta_r(\omega) \cdot {}_r\mathbf{A}_s \cdot \mathbf{S}_{ff}(\omega) \cdot {}_r\mathbf{A}_s^{*\text{T}} = \sum_{r=1}^{m < N} \mathbf{S}_{ss}^r(\omega), \quad (18)$$

where $\zeta_r(\omega)$ characterizes the dynamic part of the expression. $\mathbf{S}_{ss}^r(\omega)$ is used to mark the modal PSD stress response of the mode r and is used in the subsequent fatigue-analysis procedure.

3.2. Multiaxial criterion

The analysis procedure is now developed further, by calculating the Preumont and Piéfort [6] frequency-domain formulation of the equivalent

von Mises multiaxial criterion (EVMS) $S_c(\omega)$ for the modal stress responses $\mathbf{S}_{ss}^r(\omega)$.

$$\tilde{S}_c(\omega) = \sum_{r=1}^{m < N} S_c^r = \sum_{r=1}^{m < N} \text{Trace}[\mathbf{Q} \cdot \mathbf{S}_{ss}^r(\omega)] \quad (19)$$

where $\mathbf{Q}$ is a matrix of constant coefficients for the EVMS criterion [6]:

$$\mathbf{Q} = \begin{bmatrix} 1 & -1/2 & -1/2 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ -1/2 & -1/2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad (20)$$

The formulation in Eq. (19) is similar to the recently proposed Braccesi et al. [37] approach in that it takes the full advantage of the modal reduction procedure. However, the approach presented here preserves, by means of approximation, the respective modal contributions to damage throughout the vibration fatigue analysis procedure.

3.3. Cycle-amplitude distribution

To identify the damage, an estimate of the cycle-amplitude distribution of the equivalent stress-response PSD $S_c(\omega)$ is needed; most frequency domain methods require spectral moments [22, 42], which are thus introduced for the reduced EVMS modal stress PSD.

From Eq. (19), the spectral moment of order i for the reduced mode r is defined as:

$$m_i^r = 2 \int_0^\infty \omega^i \cdot S_c^r(\omega) d\omega. \quad (21)$$

inserting into the integral the frequency-dependent terms for the final expression:

$$m_i^r = 2 \cdot \text{Trace} \left[\mathbf{Q} \cdot {}_r\mathbf{A}_s \cdot \int_0^\infty \mathbf{S}_{ff}(\omega) \cdot \omega^i \zeta_r(\omega) d\omega \cdot {}_r\mathbf{A}_s^{*\text{T}} \right]. \quad (22)$$

It is evident that with the reduced model and under the assumption given in Eq. (16), the spectral moment i can be obtained for a particular mode r where the frequency-dependent term needs to be integrated only once (per mode). This significantly accelerates the numerical solution; without taking advantage of the modal superposition, spectral moments would require frequency integration at each material node separately. Combining that with modal reduction, reducing the number of modes from N to m , the calculation speed can be greatly increased.

The spectral moments m_i^r are summed for the total moment for the reduced modal model $\tilde{m}_i$.

$$\tilde{m}_i = \sum_{r=1}^{m < N} m_i^r \quad (23)$$

Based on the obtained spectral moments the damage identification is straightforward. Models by Dirlik [22] or Tovo-Benasciutti [23] d_{TB} were shown to give accurate results:

$$d_{\text{TB}} = \left[b + (1 - b) \alpha_2^{k-1} \right] \alpha_2 d_{\text{NB}}, \quad (24)$$

where b is a factor determined through a numerical simulation, k is the S-N slope coefficient, α_2 is the spectral width parameter and d_{NB} is the narrow-band spectral moment method [43].

3.4. Mode damage contribution

Using the expression established in (22), a measure of the damage intensity D_r can be defined that characterizes the effect a respective mode has on

the total damage intensity d . It conveys how much smaller the total damage intensity d would be if one specific mode r was absent, in relative terms.

First the partial damage intensity $d_{\setminus M_r}$ is calculated using all *but* mode r . It is then subtracted from the total damage intensity d .

$$D_r = \frac{d - d_{\setminus M_r}}{d} = 1 - \frac{d_{\setminus M_r}}{d} \quad (25)$$

By performing this procedure for each of the relevant modes within the reduced model a type of spectrum can be constructed that helps discern between the more and less damaging modes. Other measures can also be defined, *e.g.*, characterizing a change in the damping, modal constants or excitation profile.

The measure D_r can be calculated for each point with a good numerical efficiency, as all the spectral moments needed for its calculation have already been obtained in the course of the analysis. The location of the critical point might vary for different $d_{\setminus M_r}$, but the mode contribution to the damage in the total-damage d critical point will probably be of greatest interest.

4. Numerical Test Case

A finite-element method (FEM) model of a clamped beam, shown in Figure 1, is used to demonstrate the numerical efficiency of the procedure and its capability to provide additional information about the damage contribution of the respective modal shapes.

The 0.5-mm-thick steel beam is fixed at one end and excited eccentrically at the other end. A uniform unit broadband random excitation $\mathbf{S}_{ff,z}(\omega)$ is applied along the z -axis, perpendicular to the beam. A frequency range is chosen such that it excites the first five modes $r = 1, 2 \dots 5$ of the system. The

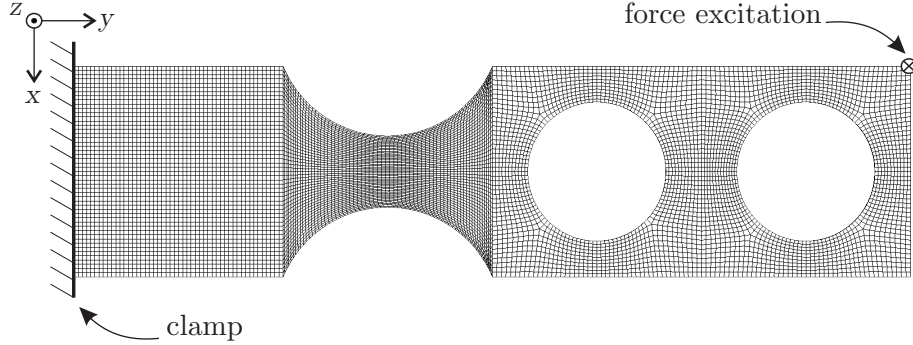


Figure 1: Finite-element model of a clamped beam.

eigenmodes are calculated for an undamped model using a commercial FEM package and are listed in Table 1. Damping factors, also given in Table 1, are applied during the modal superposition procedure.

Table 1: Excited modes of the clamped beam.

	Mode I	Mode II	Mode III	Mode IV	Mode V
ω_r [s ⁻¹]	70	405	451	1064	1746
η_r [/]	0.025	0.031	0.028	0.021	0.034

Moments of the response PSD are calculated first for each mode r , using the expression from Eq. (22), which can be slightly modified, because the excitation $\mathbf{S}_{ff,z}$ is uniform and acting only in the direction z .

$$m_i^r = 2 \cdot \underbrace{\int_0^\infty \omega^i \zeta(\omega) d\omega}_{I_i^r} \cdot \text{Trace} [\mathbf{Q} \cdot {}_r\mathbf{A}_s \cdot \mathbf{S}_{ff,z} \cdot {}_r\mathbf{A}_s^{*T}] \quad (26)$$

The integral I_i^r has to be calculated once for each mode, for each order of the spectral moment, giving stress responses across the complete numerical model. Numerical values are given for I_i^r in Table 2, where the upper limit on

the integral (frequency axis) is set to $\omega_{\max} = 5000$ rad/s. Finally, steel-type material data is picked (S-N curve parameters of $k = 3.0$ and $C = 10^{20}$) and the Tovo-Benasciutti method is used to obtain damage-intensity estimates d_{TB} . The results are shown in Figure 3.

Table 2: Values of integrals I_i^r , as calculated for each mode and each spectral moment.

	Mode I	Mode II	Mode III	Mode IV	Mode V	
I_0^r	36.864	0.152	0.122	0.012	0.002	$\times 10^{-5}$
I_1^r	25.55	0.612	0.546	0.131	0.030	$\times 10^{-3}$
I_2^r	0.800	0.125	0.124	0.070	0.026	$\times 10^{+0}$
I_4^r	1.877	5.090	6.043	16.820	16.87	$\times 10^{+4}$

Using the results of the analysis (spectral moments from Table 2), the measure D_r from Eq. (25) is calculated for each of the five modes. The results in Figure 2 show that modes I and III contribute the most to the total damage intensity. It should be noted that the measure D_r will change with respect to the modal constants, modal damping, material fatigue parameters, and the methods used in analysis (*i.e.*, the choice of the multiaxial criterion and spectral moment method formulations).

A standard (non-approximate) analysis was performed, using the expression on the left-hand side of Eq. (16). An identical group of 5 modes was considered for the reduced model. The results are shown in Figure 4 and differ only slightly from the approximated results in Figure 3, implying that the new approach can give good results with regards to the critical location and damage distribution. While the estimate of the critical point location is identical for both approaches, the presented approach underestimates the time-to-failure with a relative error of 65%. This might be viewed as a considerable error on its own, but it is actually quite reasonable compared

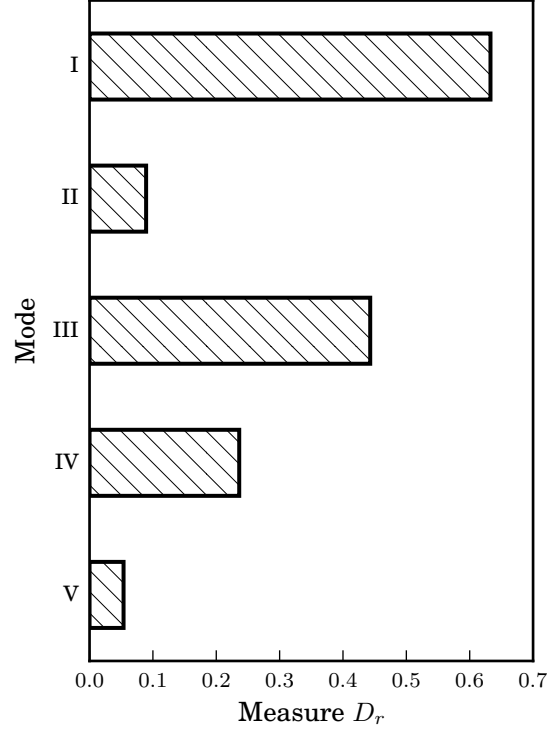


Figure 2: Comparison of the damage contribution of the specific modes.

to a scatter of up to $\pm 300\%$ that is expected in vibration-fatigue analysis [2, 9, 10].

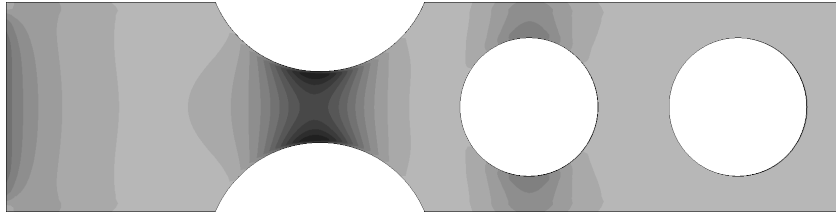


Figure 3: Damage intensity calculated for the numerical model, using the suggested approach (darker shade indicates a higher damage intensity d).

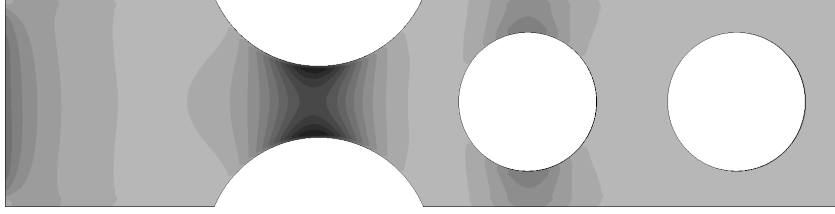


Figure 4: Damage intensity calculated for the numerical model, using standard reduced modal model approach (darker shade indicates a higher damage intensity d).

5. Discussion

The Tovo-Benasciutti spectral method was used in a numerical example that requires the following four spectral moments: m_0^r , m_1^r , m_2^r and m_4^r . In the example of the FEM beam model, comprising 18569 nodes, 18569×4 evaluations of the integral were needed, but only 5×4 evaluations were needed using the new approach. This makes the modal decomposition approach 10^3 times faster in the case of the example used here, without considering the memory requirements, which are also much lower. The speed advantage increases in proportion to the number of DOF, assuming the response frequency range does not change.

The speed advantage of the presented approach is mainly due to the combination of modal superposition and modal reduction, which gives the stress response for the complete model at each mode. By reducing the number of modes to a subset, participating within the relevant frequency range, the number of degrees of freedom is greatly reduced. The integrals needed for the spectral moment calculation are then only evaluated for each mode of the reduced model, which is computationally much more efficient compared with evaluating the stress-response spectrum for each node of the model separately.

The fact that the stress response is formulated in the form of a function of dynamic properties, as in Eq. (26), makes it possible to implement advanced numerical integration routines in a straightforward way. The characteristic shape of a modal peak is especially problematic during integration, as a very dense frequency sampling must be used to accurately describe the amplitude of the peak. In the case of the presented numerical example, with a flat excitation profile, the integral can be evaluated symbolically for a general case, giving a very precise result and also shortening the calculation times.

6. Conclusions

A modal decomposition approach to vibration-fatigue analysis is presented in this paper. It takes advantage of widely used structural dynamic analysis routines. Using modal superposition and reduction the stress-response spectrum is described in terms of modal shapes, frequencies and damping. A multiaxial criterion is used to reduce the stress response to a uniaxial equivalent and a spectral moment method is used to estimate the cycle-amplitude probability distribution.

The presented approach relates the dynamic model and the fatigue analysis in a way that propagates the dynamic properties of the model through the fatigue analysis to the final result in the form of a time-to-failure estimate. Different modes can then be compared in terms of their contribution to the total damage. By using this information, dynamic properties can be optimized to improve the durability of the structure. Additionally, the formulation allows for a straightforward analysis based on experimentally measured strain/stresses through the strain experimental modal analysis (EMA).

The results of a numerical study show that the modal decomposition approach agrees very well with the results of the standard modal superposition analysis with regards to damage distribution in the material, identifying the identical critical node. The time-to-failure estimate is conservative with an acceptable accuracy, as evaluated for the critical node. The presented approach was 10^3 faster than the standard procedure for the performed numerical study. However, the numerical efficiency is proportional to the number of DOF and might be even higher for a larger FEM model.

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